Macro II: Difference Equations

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Introduction

- Today: review linear algebra/difference equations.
- Apply to time series/macroeconomics.

A linear difference equation

- Simple first-order linear difference equation:

$$x_{t+1} = Ax_t + Cw_{t+1}$$

- We might think of x_t as a vector of states (capital, assets, etc.)
- and w_{t+1} as a vector of shocks.
- note that w_{t+1} is not known at time-t.
- Thus, a stochastic difference equation.

A linear difference equation

- Simple first-order linear difference equation:

$$x_{t+1} = Ax_t + Cw_{t+1}$$

- w_{t+1} as a vector of shocks:
  - A1: iid $w_{t+1} \sim N(0, I)$
  - A2 (A1'):

$$E[w_{t+1}|J_t] = 0$$
$$E[w_{t+1}w'_{t+1}|J_t] = I$$
$$J_t = [w_t, ..., w_1, x_0]$$

  - A3 (A1''):

$$E[w_{t+1}] = 0$$
$$E[w_t w'_{t-j}] = I \textit{ if } j = 0 \textit{ and } 0 \textit{ otherwise}$$

A linear difference equation

- Simple first-order linear difference equation:

$$x_{t+1} = Ax_t + Cw_{t+1}$$
$$y_t = Gx_t$$

- We can think of y_t as some type of measurement equation.
- This is called a state-space formulation.
- We could also think of y_t as a choice variable (more on this later).

Eigenvalues and eigenvectors

- eigenvector: the direction a system moves.
- eigenvalue: the distance it moves in that direction.
- Simple first-order linear difference equation:

$$x_{t+1} = Ax_t + Cw_{t+1}$$
$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{C} \end{bmatrix} w_{t+1}$$

- This says that a subset x_1 of the state is always at its initial value, $x_{1,t} = x_{1,0}$.
- i.e., it has a unit eigenvalue: solution of $(A_{11} - 1)x_{1,0}$ is any $x_{1,0}$.
- For this to be *covariance stationary*, the eigenvalues of $\tilde{A}$ must all be less than 1.
- i.e., the solution to $(A - \lambda I)v = 0$ is $|\lambda| < 1$ or $v = 0$ and $\lambda = 1$.

Lag operators: preliminaries

- Let **S** be a set of stochastic processes. Define the lag operator $L^n : \mathbf{S} \to \mathbf{S}$, n an integer, by

$$L^n \{X_t\}_{t=-\infty}^{\infty} = \{X_{t-n}\}_{t=-\infty}^{\infty}.$$

- Lag operator is linear

$$L(aX_t + bL^n X_t) = (aL + bL^{n+1})X_t,$$

so that lag operations can be manipulated like polynomials.

Preliminaries II

- Some geometry
- Because the lag operator is linear (everything nets out),

$$(1-\phi L^n)\left(\sum_{j=0}^{J}(\phi L^n)^j\right)X_t = \left(1-(\phi L^n)^{J+1}\right)X_t,$$

and if $(\phi L^n)^{J+1} X_t$ and $\left(\sum_{j=0}^{J}(\phi L^n)^j\right) X_t$ "converge"—which might be true even if $|\phi| > 1$—we get

$$\frac{1}{1-\phi L^n}X_t = \left(\sum_{j=0}^{\infty}(\phi L^n)^j\right)X_t,$$

the inverse of the operation $1-\phi L^n$

Preliminaries III

- Suppose $X_t = c$, $\forall t$. Then

$$L^n c = L^n X_t = c.$$

- The lag operator does not shift information sets

$$L^n E_t \left(X_{t+j} \right) = E_t \left(X_{t+j-n} \right) \neq E_{t-n} \left(X_{t+j-n} \right).$$

Linear difference equations again

- Another way to write it

$$\begin{aligned} & E_t(b_{t+1}) = \lambda b_t \\ \Leftrightarrow\ & E_t((1-\lambda L)\, b_{t+1}) = 0. \end{aligned}$$

- Rewrite this as

$$\begin{aligned} b_{t+1} &= \lambda b_t + \varepsilon_{t+1}, \\ \varepsilon_{t+1} &\equiv b_{t+1} - E_t(b_{t+1}). \end{aligned}$$

- As a forecast error, ε_t forms a martingale difference sequence, i.e.

$$E_t(\varepsilon_{t+1}) = 0$$

LEDE II

- Generalize

$$\begin{aligned} b_{t+1} - c\lambda^{t+1} &= \lambda b_t - \lambda c \lambda^t + \varepsilon_{t+1}, \\ (1 - \lambda L)\left(b_{t+1} - c\lambda^{t+1}\right) &= \varepsilon_{t+1}, \\ b_{t+1} &= c\lambda^{t+1} + \frac{1}{1 - \lambda L}\varepsilon_{t+1}, \end{aligned}$$

  where c is a constant
- Solution tells us b_t at any time, t.
- Goal: find (solve for) the set of admissible $\{\varepsilon_t\}$ and c
- Two approaches:
  - Backward solution: follow sequence from past to now to find current value.
  - Foward solution: start in future and work backwards to pin down path.

Backward solution

- If time starts at $-\infty$, the backward solution (if well-defined) is

$$b_t = c\lambda^t + \sum_{j=0}^{\infty} \lambda^j \varepsilon_{t-j}.$$

- If time starts at 0, the backward solution is

$$b_t = b_0\lambda^t + \sum_{j=0}^{t-1} \lambda^j \varepsilon_{t-j},$$

where b_0 is a (possibly) random variable

Solution set restrictions

- Initial conditions:
  - $\{\varepsilon_t\}$ and b_0 are given.
  - i.e., Perfect foresight $\varepsilon_t = 0, \forall t$
- Non-explosiveness:

$$\lim_{j\to\infty} E_t\left(b_{t+j}\right) = 0, \quad \forall t,$$

$$\sup_t V\left(b_t\right) < \infty.$$

Note that

$$\begin{aligned} E_t\left(b_{t+2}\right) &= E_t\left(E_{t+1}\left(b_{t+2}\right)\right) \\ &= E_t\left(\lambda b_{t+1}\right) = \lambda\left(\lambda b_t\right), \\ \Rightarrow E_t\left(b_{t+j}\right) &= \lambda^j b_t. \end{aligned}$$

Restrictions II

- If $|\lambda| < 1$, there are many c and $\{\varepsilon_t\}$ where the non-explosiveness conditions do not restrict
- But if $|\lambda| \geq 1$, the only admissible solution is $\varepsilon_t = c = 0$, so that $b_t = 0, \forall t$
- Because if any deviation from steady-state, will explode over time.
- Note that if $|\lambda| \geq 1$, then b_t cannot generally satisfy both an initial condition and a non-explosiveness condition

Nonhomogeneous differential equations

- Wish to solve
$$E_t\left(x_{t+1}\right)=\lambda x_t+z_t,$$
where $\{z_t\}$ is a stochastic forcing process.
- Generalize by adding a bubble term
$$\begin{aligned} E_t\left(x_{t+1}-b_{t+1}\right) &= \lambda x_t+z_t-\lambda b_t \\ \Leftrightarrow E_t\left(\left(1-\lambda L\right)\left(x_{t+1}-b_{t+1}\right)\right) &= z_t, \end{aligned}$$
where b_{t+1} is a "bubble term" that solves
$$E_t\left(b_{t+1}\right)=\lambda b_t.$$
- i.e., a process unrelated to the fundamental term, x_t.

General LEDE II

- The general problem is

$$
\begin{aligned}
x_{t+1} - b_{t+1} &= \lambda \left(x_t - b_t \right) + \widetilde{\eta}_{t+1} + z_t, \\
\widetilde{\eta}_{t+1} &\equiv \left(x_{t+1} - b_{t+1} \right) - E_t \left(x_{t+1} - b_{t+1} \right), \\
\left(1 - \lambda L \right) \left(x_{t+1} - b_{t+1} \right) &= \widetilde{\eta}_{t+1} + z_t. \qquad \text{(GP)}
\end{aligned}
$$

- Goal: find the set of admissible $\{\widetilde{\eta}_t\}$ and $\{b_t\}$
- $\tilde{\eta}_{t+1}$: expectational errors.
- b_t: bubble term (non-fundamental value).

Backward solution

- $\{\widetilde{\eta}_t\}$ and $\{b_t\}$ cannot be identified separately
- If time starts at $-\infty$, backward solution (if well-defined) is

$$\begin{aligned} x_{t+1} &= \sum_{j=0}^{\infty} \lambda^j \left(z_{t-j} + \widetilde{\eta}_{t+1-j} \right) + b_{t+1} \\ &= \sum_{j=0}^{\infty} \lambda^j z_{t-j} + \widetilde{b}_{t+1}, \\ \widetilde{b}_{t+1} &\equiv b_{t+1} + \sum_{j=0}^{\infty} \lambda^j \widetilde{\eta}_{t+1-j}. \end{aligned}$$

- $\widetilde{b}_{t+1}$ is a bubble term
- Fundamental (sometimes called particular) solution is

$$x_{t+1} = \sum_{j=0}^{\infty} \lambda^j z_{t-j}$$

- i.e., must reflect sequence of shocks (stochastic forcing process).

Backwards solution II

- If time starts at 0, the backward solution can be written as

$$x_{t+1} = \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \widetilde{\eta}_{t+1-j} + (x_0 - b_0)\,\lambda^{t+1} + b_{t+1},$$

which becomes

$$\begin{aligned} x_{t+1} &= \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \eta_{t+1-j} + x_0 \lambda^{t+1}, \\ \eta_t &\equiv \widetilde{\eta}_t + b_t - E_{t-1}(b_t) \\ &= x_t - E_{t-1}(x_t). \end{aligned}$$

- x_t is stochastic, will have errors.
- b_t is deterministic. Cannot be wrong or will be systematic.

Forward solution

- First, rewrite

$$(1-\lambda L)(x_{t+1}-b_{t+1})=\widetilde{\eta}_{t+1}+z_t$$

as

$$\left(\frac{1-\lambda L}{-\lambda L}\right)(-\lambda L)(x_{t+1}-b_{t+1})=\widetilde{\eta}_{t+1}+z_t,$$

$$\left(1-\lambda^{-1}L^{-1}\right)(x_t-b_t)=-\frac{1}{\lambda}(z_t+\widetilde{\eta}_{t+1}).$$

To ensure that x_t is a function only of variables known at time t, write this as

$$E_t\left(\left(1-\lambda^{-1}L^{-1}\right)(x_t-b_t)\right)=-\frac{1}{\lambda}E_t(z_t+\widetilde{\eta}_{t+1}).$$

Forward solution II

- Invert the lag operator

$$E_t\left(x_t - b_t\right) = -\frac{1}{\lambda}E_t\left(\frac{1}{1-\lambda^{-1}L^{-1}}\left(z_t + \widetilde{\eta}_{t+1}\right)\right),$$

$$\begin{aligned} x_t &= -\frac{1}{\lambda}E_t\left(\sum\nolimits_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^j\left(z_{t+j} + \widetilde{\eta}_{t+j+1}\right)\right) + b_t, \\ &= -\frac{1}{\lambda}E_t\left(\sum\nolimits_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^j z_{t+j}\right) + b_t, \end{aligned}$$

  because $E_t\left(\widetilde{\eta}_{t+j}\right) = 0, \forall j \geq 1$
- note: $\frac{1}{L}^j = L^{-j}$ subsumed into z_{t+j} (bc negative exponent on lag operator equals lead operator)

Forward solution III

- The fundamental (particular) solution is

$$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right)$$

- Note that $\widetilde{\eta}_t$ depends only on the forcing process z_t

$$\widetilde{\eta}_t = -\frac{1}{\lambda} \left[E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right) \right. \\ \left. - E_{t-1} \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right) \right], \forall t.$$

Summing up

- Forward solution

$$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j z_{t+j} \right) + b_t.$$

- Backward solution

$$x_{t+1} = \sum_{j=0}^{\infty} \lambda^j z_{t-j} + \tilde{b}_{t+1},$$

or

$$x_{t+1} = \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \eta_{t+1-j} + x_0 \lambda^{t+1}.$$

Restrictions

- Initial conditions:
  - x_0 and $\{\widetilde{\eta}_t\}_{t=1}^{\infty}$ are directly given, for example with capital accumulation

$$
\begin{aligned}
& k_{t+1} = (1-\delta)\, k_t + i_t, \\
& k_0 \text{ given}, \\
& k_{t+1} - E_t\left(k_{t+1}\right) = 0,\ \forall t.
\end{aligned}
$$

- Non-Explosiveness (boundary condition):

$$
\begin{aligned}
& \lim_{j \to \infty} E_t\left(x_{t+j}\right) = 0, \quad \forall t, \\
& \sup_t V\left(x_t\right) < \infty.
\end{aligned}
$$

Solutions

- If $|\lambda| < 1$, for "well-behaved" $\{z_t\}$ (e.g, ARMA processes), one solves $(1-\lambda L)^{-1}$ backwards to get

$$x_{t+1} = \sum_{j=0}^{\infty} \lambda^j z_{t-j} + \tilde{b}_{t+1},$$

  with a large number of permissable $\left\{\tilde{b}_t\right\}$.

- But if $|\lambda| > 1$, for "typical" $\{z_t\}$ (e.g, ARMA processes), we must solve $(1-\lambda L)^{-1}$ forward and set $b_t = 0$, so that

$$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j z_{t+j} \right).$$

- If $|\lambda| > 1$, cannot satisfy both initial conditions and non-explosiveness

Rule of Thumb

- If $|\lambda| < 1$, set

$$x_{t+1} = \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \eta_{t+1-j} + x_0 \lambda^{t+1}.$$

  and use initial conditions to pin down x_0 and $\{\eta_t\}$
- If $|\lambda| > 1$, set

$$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right).$$

- If $|\lambda| = 1$, consider case by case

Next Time

- Discuss rational expectations and Lucas Critique.
- See my webpage for new homework.