Instructor: *Professor Griffy*
Due: *Mar., 22nd 2022*
AECO 701

Problem Set 4

Income Fluctuations with CARA Utility

Problem 1. Solving for Consumption. You're asked to study an optimal savings plan when households face fluctuating income. The exponential (or CARA) utility function is tractable and it allows for closed-form solutions using a guess-and-verify method. Consider an agent with the following utility maximization problem:

$$\mathbb{E}\sum_{t=1}^{\infty}\left(\frac{1}{1+\delta}\right)^{t}u(c_t) \tag{1}$$

subject to

$$y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0,\sigma) \tag{2}$$

$$\delta > 0, \qquad 0 < \phi < 1, \tag{3}$$

where utility takes the CARA form $u(c) = -\frac{1}{\theta}e^{-\theta c}$.

1. The recursive formulation of this problem is given by

$$V(A,y) = \max_{c}\left\{u(c) + \beta\mathbb{E}[V(A',y')]\right\} \tag{4}$$

$$\text{s.t.} \qquad A' = (1+r)A + y - c. \tag{5}$$

   Take the first-order condition in consumption and solve for the within period relationship between assets and consumption.

2. Guess that the value function takes the form

$$V(A,y) = -\frac{1}{\theta r}e^{-\theta r(A+ay+\bar{b})}. \tag{6}$$

   Using the relationship you derived in part (a), show that the candidate optimal consumption rule takes the form

$$c^* = r(A + ay + a_0), \tag{7}$$

   where we define

$$a_0 = \bar{b} + \frac{1}{\theta r}ln(1+r). \tag{8}$$

   Note that $a = \frac{1}{1+r-\phi_1}$, which means that ay is the present value of human wealth given by

$$h_t = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}y_t = \frac{y_t}{1+r-\phi_1}. \tag{9}$$

3. Using our guess of the value function, we can rewrite the Bellman Equation as

$$V(A,y) = \frac{r}{1+r}V(A,y) - \left(\frac{1}{1+\delta}\right)\frac{1}{\theta r}\mathbb{E}\left[exp\left(-\theta r\left(A' + ay' + \bar{b}\right)\right)\right]. \tag{10}$$

Plug in the equation for the evolution of assets for A' and the AR(1) process that determines income for y', as well as your guess for V, and show that consumption is equal to

$$c = r\left\{A + \frac{1 - a + a\phi_1}{r}y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[ln\left(\frac{1+\delta}{1+r}\right) - \ln\left(\mathbb{E}\left[exp\left(-\theta r a\varepsilon'\right)\right]\right)\right]\right\}. \tag{11}$$

(Two hints: 1. Derivatives are not required!; 2. Remember that $exp(a+b) = exp(a) \times exp(b)$)

4. Using the method of undetermined coefficients (aka guess and verify - set your two solutions for consumption equal), solve for $\bar{b}$ using your solution obtained in part (b).

5. Show that this solution for consumption can be written as

$$c^* = r(A + h - \Gamma(r)), \tag{12}$$

where $h = a(y + \frac{\phi_0}{r})$ is human wealth and $\Gamma(r) = \frac{1}{\theta r^2}[ln(\mathbb{E}[exp(-\theta r a\varepsilon')]) - ln(\frac{1+\delta}{1+r})]$ is the difference between precautionary savings and impatience caused by a distaste for lower consumption.