Instructor: *Professor Griffy*
Due: *Mar., 22nd 2022*
AECO 701

Problem Set 4

Income Fluctuations with CARA Utility You're asked to study an optimal savings plan when households face fluctuating income. The exponential (or CARA) utility function is tractable and it allows for closed-form solutions using a guess-and-verify method. Consider an agent with the following utility maximization problem:

$$\mathbb{E}\sum_{t=1}^{\infty}\left(\frac{1}{1+\delta}\right)^{t}u(c_t) \tag{1}$$

subject to

$$y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0,\sigma) \tag{2}$$

$$\delta > 0, \qquad 0 < \phi < 1, \tag{3}$$

where utility takes the CARA form $u(c) = -\frac{1}{\theta}e^{-\theta c}$.

- The recursive formulation of this problem is given by

$$V(A,y) = \max_{c}\left\{u(c) + \beta\mathbb{E}[V(A',y')]\right\} \tag{4}$$

$$\text{s.t.} \qquad A' = (1+r)A + y - c. \tag{5}$$

  Take the first-order condition in consumption and solve for the within period relationship between assets and consumption.

  We can construct:

$$\mathcal{L}(A,y) = -\frac{1}{\theta}e^{-\theta c} + \beta\mathbb{E}[V(A',y')] + \lambda[(1+r)A - A' + y - c].$$

  And then taking the F.O.C. w.r.t. consumption:

$$\frac{\partial\mathcal{L}}{\partial c} = e^{-\theta c} - \lambda = 0 \Longrightarrow \lambda = e^{-\theta c}.$$

  We also know the marginal value obtained from current assets:

$$\frac{\partial\mathcal{L}}{\partial A} = \lambda(1+r).$$

  Thus combining the two we obtain a relationship between assets and consumption within a period:

$$\boxed{\frac{\partial\mathcal{L}}{\partial A} = (1+r)e^{-\theta c}.}$$

- Guess that the value function takes the form

$$V(A,y) = -\frac{1}{\theta r} e^{-\theta r(A+ay+\bar{b})}. \tag{6}$$

Using the relationship you derived in part (a), show that the candidate optimal consumption rule takes the form

$$c^* = r(A + ay + a_0), \tag{7}$$

where we define

$$a_0 = \bar{b} + \frac{1}{\theta r} ln(1+r). \tag{8}$$

Note that $a = \frac{1}{1+r-\phi_1}$, which means that ay is the present value of human wealth given by

$$h_t = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t = \frac{y_t}{1+r-\phi_1}. \tag{9}$$

First, we can invoke the envelope theorem to say that $\frac{\partial \mathcal{L}}{\partial A} = \frac{\partial V}{\partial A}$. That is, using our guess in (6), we find that

$$\frac{\partial \mathcal{L}}{\partial A} = e^{-\theta r(A+ay+\bar{b})}.$$

We can then plug this into our answer in part (a) and do some algebra...

$$\begin{aligned}
e^{-\theta r(A+ay+\bar{b})} &= (1+r)e^{-\theta c} \\
-\theta r(A + ay + \bar{b}) &= -\theta c + ln(1+r) && \text{(taking log of both sides)} \\
c^* &= r(A + ay + \bar{b}) + \frac{1}{\theta} ln(1+r) && \text{(solving for } c^*\text{)} \\
&= r(A + ay + \bar{b} + \frac{1}{\theta r} ln(1+r)) \\
&= \boxed{r(A + ay + a_0)} && \text{(plugging in for } a_0\text{)}
\end{aligned}$$

- Using our guess of the value function, we can rewrite the Bellman Equation as

$$V(A,y) = \frac{r}{1+r} V(A,y) - \left(\frac{1}{1+\delta}\right) \frac{1}{\theta r} \mathbb{E}\left[exp\left(-\theta r\left(A' + ay' + \bar{b}\right)\right)\right]. \tag{10}$$

Plug in the equation for the evolution of assets for A' and the AR(1) process that determines income for y', as well as your guess for V, and show that consumption is equal to

$$c = r\left\{A + \frac{1-a+a\phi_1}{r} y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[ln\left(\frac{1+\delta}{1+r}\right) - \ln\left(\mathbb{E}\left[exp\left(-\theta r a \varepsilon'\right)\right]\right)\right]\right\}. \tag{11}$$

(Two hints: 1. Derivatives are not required!; 2. Remember that $exp(a+b) = exp(a) \times exp(b)$)

We start by subtracting the $V(A,y)$ term on the right over to get it to simplify *slightly*.

$$\begin{aligned}
&\left(\frac{1}{1+r}\right)\left(\frac{-1}{\theta r}\right) exp\{-\theta r(A + ay + \bar{b})\} \\
&\qquad = \left(\frac{1}{1+\delta}\right)\left(\frac{-1}{\theta r}\right) \mathbb{E}\left[exp\left\{-\theta r\left[(1+r)A + y - c + a(\phi_0 + \phi_1 y + \varepsilon') + \bar{b}\right]\right\}\right]
\end{aligned}$$

Next, note that the expectation term simplifies. We can pull out things that are already determined (i.e. things without primes on them):

$$\Longrightarrow exp\left\{-\theta r\left[(1+r)A + y - c + a\phi_0 + a\phi_1 y + \bar{b}\right]\right\}\mathbb{E}\left[exp\left\{-\theta r a\varepsilon'\right\}\right].$$

Now we can take the ln of both sides:

$$-\theta r(A + ay + \bar{b}) - \ln\left(\frac{1+\delta}{1+r}\right) = -\theta r[(1+r)A + y - c + a\phi_0 + a\phi_1 y + \bar{b}] + \ln(\mathbb{E}[exp\{-\theta r a\varepsilon'\}]).$$

Isolating the c term:

$$\theta r c = \theta r[(1+r)A + y + a\phi_0 + a\phi_1 y + \bar{b}] - \theta r(A + ay + \bar{b}) + \ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp\{-\theta r a\varepsilon'\}]).$$

Dividing by θr and simplifying:

$$c = rA + (1 - a + a\phi_1)y + a\phi_0 + \frac{1}{\theta r}\left[ln\left(\frac{1+\delta}{1+r}\right) - \ln\left(\mathbb{E}\left[exp\left\{-\theta r a\varepsilon'\right\}\right]\right)\right].$$

$$\Longrightarrow \boxed{c = r\left\{A + \frac{1-a+a\phi_1}{r}y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[ln\left(\frac{1+\delta}{1+r}\right) - \ln\left(\mathbb{E}\left[exp\left(-\theta r a\varepsilon'\right)\right]\right)\right]\right\}}.$$

- Using the method of undetermined coefficients (aka guess and verify - set your two solutions for consumption equal), solve for $\bar{b}$ using your solution obtained in part (b).

First we take our first solution for c given by (7) and plug in for a_0 given by (8). Then, setting this equal to our last solution for consumption:

$$\cancel{r(A}+ay+\bar{b}+\frac{1}{\theta r}\ln(1+r))$$
$$= \cancel{r\{A} + \frac{1-a+a\phi_1}{r}y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}[\ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a\varepsilon')])]\}$$

Grouping terms together:

$$\bar{b} = \left(\frac{1-a+a\phi_1}{r} - a\right)y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[\ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a\varepsilon')]) - r\ln(1+r)\right]$$

$$= \underbrace{\left(\frac{1-a\overbrace{(1+r-\phi_1)}^{=1/a}}{r}\right)}_{=0}y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[\ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a\varepsilon')]) - r\ln(1+r)\right]$$

$$\Longrightarrow \boxed{\bar{b} = \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a\varepsilon')]) - rln(1+r)\right]}$$

- Show that this solution for consumption can be written as

$$c^* = r(A + h - \Gamma(r)), \tag{12}$$

where $h = a(y + \frac{\phi_0}{r})$ is human wealth and $\Gamma(r) = \frac{1}{\theta r^2}[ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - ln(\frac{1+\delta}{1+r})]$ is the difference between precautionary savings and impatience caused by a distaste for lower consumption.

Recall that our expression for consumption was given by $c = r(A + ay + \bar{b} + \frac{1}{\theta r} ln(1+r))$ (plug (8) into (7)). Plugging what we found for $\bar{b}$ and simplifying:

$$c^* = r\{A + ay + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[\ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - \cancel{r\ln(1+r)}\right] + \cancel{\frac{1}{\theta r}\ln(1+r)}\}$$

$$= r\Big\{A + \underbrace{a\left(y + \frac{\phi_0}{r}\right)}_{h} - \underbrace{\frac{1}{\theta r^2}\left[\ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - \ln\left(\frac{1+\delta}{1+r}\right)\right]}_{\Gamma(r)}\Big\}$$

$$\Longrightarrow \boxed{c^* = r(A + h - \Gamma(r))}$$