Macro II

Professor Griffy

UAlbany

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Announcements

- Today: Start discussing solution techniques.
- Focus on linearization & its problems.
- New homework on my website.

Motivation

- Models are hard to solve globally.
- Requires a lot of grid points, entails curse of dimensionality, takes a long time.
- A linearized system, by contrast, is easy to solve.
- Need to pick a place to linearize around.
- Pick the steady state.
- Underlying assumption: economy will stay close to the steady-state.

Empirical Motivation

- Standard RBC: all fluctuations of hours worked on the *intensive* margin, i.e. average number of hours worked.
- Data: little fluctuation in average hours worked; lots of fluctuation in whether or not people are working (*extensive* margin).
- Standard RBC: missed badly on labor fluctuations (Frisch Elasticity, i.e. response of labor to change in wage too low).
- Solution: Modify model to have extensive margin with high Frisch Elasticity.
- Now: households pick the *probability* of working, but have to work a set number of hours.
- This is a *nonconvexity* in that it forces individuals to work either 0 or h hours.

Hansen (1985)

- Neoclassical growth model with labor-leisure lottery.
- A social planner maximize the following:

$$E(\sum_{t=0}^{\infty} \beta^t [ln(C_t) - \gamma H_t] \tag{1}$$

- Subject to the following constraints:

$$Y_t = A_t K_t^{\theta} (\eta^t H_t)^{1-\theta} \tag{2}$$

$$ln(A_t) = (1-\rho) ln(A) + \rho ln(A_{t-1}) + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2) \tag{3}$$

- The goods market clears and capital evolves in a predetermined fashion.
- Here, we assume that per capita labor productivity grows at rate η.

Equilibrium

- First step: detrend appropriate variables by per capita growth to get stationarity: i.e. $y_t = Y_t/\eta^t$.
- The system of equations that characterize the equilibrium are:

$$y_t = a_t k_t^{\theta} h_t^{1-\theta} \tag{4}$$

$$ln(a_t) = (1-\rho)ln(A) + \rho ln(a_{t-1}) + \epsilon_t \tag{5}$$

$$y_t = c_t + i_t \tag{6}$$

$$\eta k_{t+1} = (1-\delta)k_t + i_t \tag{7}$$

- Combine FOC[c] and FOC[h]:

$$\gamma c_t h_t = (1-\theta)y_t \tag{8}$$

- Euler Equation:

$$\frac{\eta}{c_t} = \beta E_t[\frac{1}{c_{t+1}}(\theta(\frac{y_{t+1}}{k_{t+1}}) + 1 - \delta)] \tag{9}$$

Solving for the Steady-State

$$ln(a^*) = (1-\rho)ln(A) + \rho ln(a^*)$$
$$\Rightarrow ln(a^*) = ln(A) \qquad (10)$$

Euler Equation:

$$\frac{\eta}{c^*} = \beta E_t[\frac{1}{c^*}(\theta(\frac{y^*}{k^*}) + 1 - \delta)]$$

$$\Rightarrow \frac{\eta}{\beta} = \theta\frac{y^*}{k^*} + 1 - \delta$$

$$\Rightarrow k^* = (\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})y^* \qquad (11)$$

Solving for the Steady-State

- Use the previous to solve for investment

$$\eta k^* = (1-\delta)k^* + i^*$$

$$\Rightarrow (\eta - 1 + \delta)k^* = i^*$$

$$\Rightarrow i^* = \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)y^* \tag{12}$$

- FOC[c] and FOC[h]:

$$\gamma c^* h^* = (1-\theta)y^*$$

$$\Rightarrow \gamma\left[1 - \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)\right]y^* h^* = (1-\theta)y^*$$

$$\Rightarrow h^* = \left(\frac{1-\theta}{\gamma}\right)\left[1 - \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)\right]^{-1} \tag{13}$$

Solving for the Steady-State

- Finally, solve for output.

$$y^* = a^* k^{*\theta} h^{*1-\theta}$$

$$y^* = a^*((\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})y^*)^\theta [(\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1}]^{1-\theta}$$

$$y^{*1-\theta} = a^*(\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})^\theta [(\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1}]^{1-\theta}$$

$$y^* = a^{*\frac{1}{1-\theta}}(\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})^{\frac{\theta}{1-\theta}} [(\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1}]^{1-\theta} \tag{14}$$

- All variables now a function of parameters.

Steady-States

- In steady-state $y_t = y_{t+1} = y^*$.

$$ln(a^*) = ln(A) \tag{15}$$

$$k^* = (\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})y^* \tag{16}$$

$$i^* = (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})y^* \tag{17}$$

$$c^* = [1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]y^* \tag{18}$$

$$h^* = (\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1} \tag{19}$$

$$y^* = a^{*\frac{1}{1-\theta}}(\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})^{\frac{\theta}{1-\theta}}[(\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1}]^{1-\theta} \tag{20}$$

- These steady-states will be used for calibration/solving.

Overview

- Broadly, two methods of solving models:
  1. Local linear methods.
  2. Global non-linear methods.
- Tradeoff: accuracy (global non-linear) for speed and simplicity (local linear).
- My preference: global methods (linear methods involve linearizing Euler Equation, distorting choices over risk).
- Here: Discuss log linearization and Blanchard and Kahn's Method.

Local Linear Methods

- Log-linearize the system around the steady-state, then proceed.
- First have to solve the system for stability:
  1. Klein's Method (2000): Used for singular matrices.
  2. Sim's Method (2001): Used when it is unclear which variables are states and controls.
  3. Blanchard and Kahn's Method (1980): First solution method for rational expectations models.
- Here, we will use Blanchard and Kahn's Method.

Log-Linearizing the System

We first wish to rewrite $\tilde{x}_t = ln(x_t) - ln(x)$ in two convenient ways:

$$\tilde{x}_t = ln(\frac{x_t}{x})$$

Then, the first-order Taylor Approximation to this equation yields:

$$\tilde{x}_t \approxeq \tilde{x}_t(x) + \frac{\partial \tilde{x}_t}{\partial x_t}(x)(x_t - x)$$

$$\Rightarrow \tilde{x}_t \approxeq ln(1) + \frac{1}{x}(x_t - x)$$

We can also rewrite the equation for $\tilde{x}_t$ as

$$x_t = xe^{\tilde{x}_t} \tag{21}$$

Log-Linearizing the System

From equilibrium conditions:

$$y_t = a_t k_t^{\theta} h_t^{1-\theta} \tag{22}$$

$$\Rightarrow ln(y_t) = ln(a_t) + \theta ln(k_t) + (1-\theta) ln(h_t)$$

$$ln(y) = ln(a) + \theta ln(k) + (1-\theta) ln(h)$$

$$\begin{aligned} \Rightarrow \tilde{y}_t = ln(y_t) - ln(y) = ln(a_t) \quad & + \quad \theta ln(k_t) + (1-\theta) ln(h_t) \\ & - \quad (ln(a) + \theta ln(k) + (1-\theta) ln(h)) \end{aligned}$$

$$\Rightarrow \tilde{y}_t = \tilde{a}_t + \theta \tilde{k}_t + (1-\theta) \tilde{h}_t \tag{23}$$

Log-Linearizing the System

$$ln(a_t) = (1-\rho)ln(A) + \rho ln(a_{t-1}) + \epsilon_t$$

$$ln(a) = (1-\rho)ln(A) + \rho ln(a)$$

$$\Rightarrow \tilde{a}_t = \rho \tilde{a}_{t-1} + \epsilon_t \tag{24}$$

Log-Linearizing the System

$$y_t = c_t + i_t$$

$$\Rightarrow \tilde{x}_t \approxeq ln(1) + \frac{1}{x}(x_t - x) = (\frac{x_t}{x} + 1)$$

$$\Rightarrow y(\tilde{y}_t + 1) = c(\tilde{c}_t + 1) + i(\tilde{i}_t + 1)$$

$$\tilde{y}_t = \frac{c}{y}\tilde{c}_t + \frac{i}{y}\tilde{i}_t$$

Log-Linearizing the System

- Let $\tilde{y}_t = ln(y_t) - ln(y^*)$. Then, using Taylor Series approximations, the system characterizing the equilibrium becomes:

$$\tilde{y}_t = \tilde{a}_t + \theta\tilde{k}_t + (1-\theta)\tilde{h}_t \quad (25)$$

$$\tilde{a}_t = \rho\tilde{a}_{t-1} + \epsilon_t \quad (26)$$

$$(\frac{\eta}{\beta} - 1 + \delta)\tilde{y}_t = [\frac{\eta}{\beta} - 1 + \delta - \theta(\eta - 1 + \delta)]\tilde{c}_t + \theta(\eta - 1 + \delta)\tilde{i}_t \quad (27)$$

$$\eta\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + (\eta - 1 + \delta)\tilde{i}_t \quad (28)$$

$$\tilde{y}_t = \tilde{c}_t + \tilde{h}_t \quad (29)$$

$$0 = \frac{\eta}{\beta}\tilde{c}_t + E[(\frac{\eta}{\beta} - 1 + \delta)(\tilde{y}_{t+1} - \tilde{k}_{t+1}) - \frac{\eta}{\beta}\tilde{c}_{t+1}] \quad (30)$$

Log-Linearizing the System

- We can now write the system as:

$$\Psi_1 \zeta_t = \Psi_2 \xi_t + \Psi_3 \tilde{a}_t \tag{ME}$$

$$\Psi_4 E_t(\xi_{t+1}) = \Psi_5 \xi_t + \Psi_6 \zeta_t + \Psi_7 \tilde{a}_t \tag{TE}$$

- ζ_t are static predetermined and nonpredetermined variables, $[\tilde{y}_t, \tilde{i}_t, \tilde{h}_t]'$.
- ξ_t are dynamic predetermined and nonpredetermined variables, $[\tilde{k}_t, \tilde{c}_t]'$.
- $\tilde{a}_t$ is the technology process.
- Why is $\tilde{c}_t$ among the dynamic variables?

Matrices

$$\kappa = \eta/\beta - 1 + \delta$$

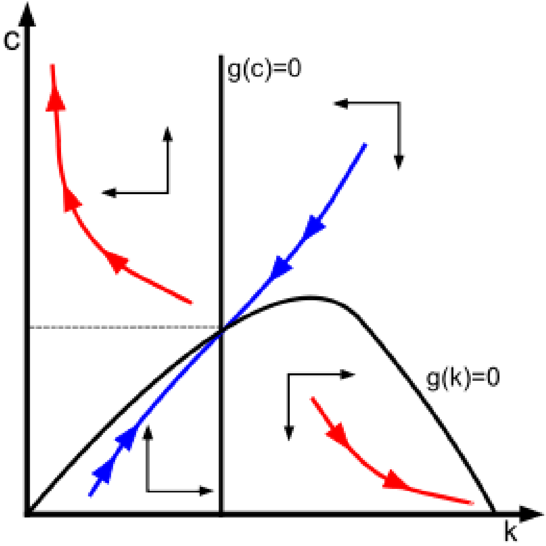
$$\lambda = \eta - 1 + \delta$$

$$\zeta_t = \begin{bmatrix} \tilde{y}_t & \tilde{i}_t & \tilde{h}_t \end{bmatrix}', \quad \xi_t = \begin{bmatrix} \tilde{k}_t & \tilde{c}_t \end{bmatrix}'$$

$$\Psi_1 = \begin{bmatrix} 1 & 0 & \theta - 1 \\ \kappa & -\theta\lambda & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} \theta & 0 \\ 0 & \kappa - \theta\lambda \\ 0 & 1 \end{bmatrix}, \quad \Psi_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Psi_4 = \begin{bmatrix} \eta & 0 \\ \kappa & \eta/\beta \end{bmatrix}, \quad \Psi_5 = \begin{bmatrix} 0 & 0 & 0 \\ -\kappa & 0 & 0 \end{bmatrix}, \quad \Psi_6 = \begin{bmatrix} 1 - \delta & 0 \\ 0 \eta/\beta & \end{bmatrix}, \quad \Psi_7 = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving the Model - Blanchard and Kahn (1980)



- Select $\tilde{c}_0$ st the system isn't explosive (optimal control!).

Solving the Model - Cont.

- Solve systems (TE and ME) so that ξ_{t+1} is only a function on ξ_t and $\tilde{a}_t$:

$$\Psi_1 \zeta_t = \Psi_2 \xi_t + \Psi_3 \tilde{a}_t \tag{31}$$

$$\Psi_4 E_t(\xi_{t+1}) = \Psi_5 \xi_t + \Psi_6 \zeta_t + \Psi_7 \tilde{a}_t \tag{32}$$

$$\Rightarrow \zeta_t = \Psi_1^{-1}[\Psi_2 \xi_t + \Psi_3 \tilde{a}_t]$$

- Plug into transition equation:

$$\Psi_4 E_t(\xi_{t+1}) = \Psi_5 \xi_t + \Psi_6 \Psi_1^{-1}[\Psi_2 \xi_t + \Psi_3 \tilde{a}_t] + \Psi_7 \tilde{a}_t$$

$$\Rightarrow E_t(\xi_{t+1}) = \Psi_4^{-1}[\Psi_5 + \Psi_6 \Psi_1^{-1} \Psi_2] \xi_t + \Psi_4^{-1}[\Psi_7 + \Psi_6 \Psi_1^{-1} \Psi_3] \tilde{a}_t \tag{33}$$

- Desired result!

Solving the Model - Cont.

- Having solved systems on previous slide so that ξ_{t+1} is only a function on ξ_t and $\tilde{a}_t$:

$$\begin{bmatrix} \tilde{k}_{t+1} \\ E_t(\tilde{c}_{t+1}) \end{bmatrix} = \Lambda^{-1} J \Lambda \begin{bmatrix} \tilde{k}_t \\ \tilde{c}_t \end{bmatrix} + E\tilde{a}_t \tag{34}$$

- $\Lambda^{-1} J \Lambda$ is the Jordan Decomposition.
- Subsume Λ into the model variables, denoted by hats:

$$\hat{c}_t = \Lambda_{12}\tilde{k}_t + \Lambda_{22}\tilde{c}_t \tag{35}$$

Solving the Model - Cont.

- Subsume Λ into the model variables, denoted by hats.

$$\begin{bmatrix} \hat{k}_{t+1} \\ E_t(\hat{c}_{t+1}) \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + D\tilde{a}_t \tag{36}$$

$$E_t(\hat{c}_{t+1}) = J_2\hat{c}_t + D_2\tilde{a}_t \tag{37}$$

- $J_2 > 1 \rightarrow$ bad choice of c_t and this explodes.
- Solution: pick c_t so that it isn't a function of c_{t-1}!
- Rearranging:

$$\hat{c}_t = J_2^{-1}E_t(\hat{c}_{t+1}) - J_2^{-1}D_2\tilde{a}_t \tag{38}$$

Solving the Model - Cont.

- Iterating on previous equation:

$$\hat{c}_{t+1} = J_2^{-1} E_t(\hat{c}_{t+2}) - J_2^{-1} D_2 \tilde{a}_{t+1} \tag{39}$$

$$\Rightarrow \hat{c}_t = J_2^{-1} E_t(J_2^{-1} E_t(\hat{c}_{t+2}) - J_2^{-1} D_2 \tilde{a}_{t+1}) - J_2^{-1} D_2 \tilde{a}_t$$

$$\Rightarrow \hat{c}_t = J_2^{-2} E_t(\hat{c}_{t+2})) - J_2^{-2} D_2 \rho \tilde{a}_t - J_2^{-1} D_2 \tilde{a}_t \tag{40}$$

- Impose transversality condition (i.e. $E_t(\hat{c}_{t+i})) = 0$ for large enough i):

$$\Rightarrow \hat{c}_t = -\sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 \rho \tilde{a}_t \tag{41}$$

Solving the Model - Cont.

- Iterating on (33):

$$\hat{c}_t = \Lambda_{12}\tilde{k}_t + \Lambda_{22}\tilde{c}_t$$

$$\Rightarrow \Lambda_{22}\tilde{c}_t = -\Lambda_{12}\tilde{k}_t - \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 \rho \tilde{a}_t$$

- Solving this yields:

$$\Rightarrow c_t = -\Lambda_{22}^{-1}\Lambda_{12}\tilde{k}_t + (1/\Lambda_{22})(\frac{D_2}{\rho - J_2})\tilde{a}_t \qquad (42)$$

- The system will now be saddle-path stable.

Next Time

- Value function iteration.
- See my website for homework.