Quantitative Macro-Labor: Heterogeneous Agent Models

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Fall 2022

Announcements

- Today: Solving heterogeneous agent models.
- Final project: write down and solve a model to explain your empirical regularity.
- Empirical regularities project due next Tuesday.
- No class next week (but project still due!)

Heterogeneous Agent Production Economy

- In a production economy, the agent's problem is given by

$$
\begin{aligned}
V(k,\epsilon;\psi) &= u(c) + \beta E[V(k'\epsilon';\psi')] && (1)\\
\text{s.t. } c + k' &\leq (1 + r(K,L) - \delta)k + w(K,L)\epsilon && (2)\\
k' &\geq \underline{k} && (3)\\
\epsilon &\sim \text{ Markov } P(\epsilon'|\epsilon) && (4)\\
\psi' &= \Psi(\psi) && (5)\\
c \geq 0, k &\geq 0, k_0 \text{ given} && (6)
\end{aligned}
$$

- ϵ is a markov process that yields hours worked.
- Ψ is an unspecified evolution of the aggregate state (k,ϵ).
- Prices are determined from the firm's problem

Prices - The Firm's Problem

- How we handle prices determines the difficulty of this problem.
- In this economy, a single firm produces using labor (hours) and capital.

$$\Pi = \max_{K,L} F(K,L) - wL - rK \tag{7}$$

- This yields standard competitive prices for the rental rates.

Stationary Recursive Competitive Equilibrium

- A stationary RCE is given by pricing functions r, w, a worker value function $V(k, \epsilon; \psi)$, worker decision rules k', c, a type-distribution $\psi(k, \epsilon)$, and aggregates K and L that satisfy
  1. k' and c are the optimal solutions to the worker's problem given prices.
  2. Prices are formed competitively from the firm's problem.
  3. Consistency between aggregate evolution and individual decision rules: ψ is the stationary distribution implied by worker decision rules.
  4. Aggregates are consistent with individual policy rules: $K = \int k d\psi$, $L = \int \epsilon d\psi$

Calibration

- Functions:
  - Utility: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$
  - Production: $F(K, L) = K^{\alpha} L^{1-\alpha}$
- Borrowing constraint: $\underline{k} = 0$
- $\alpha = 0.36$.

Solving the Model: Market Clearing

- In equilibrium

$$K = \sum_k \sum_\epsilon k_s(k, \epsilon)\psi(k, \epsilon) \tag{8}$$

- where k_s is the supply of savings.
- What must the equilibrium prices satisfy?

$$r = F_K(K_D, L) \tag{9}$$
$$K_D(r) = K_S(r) \tag{10}$$

- Fixing K_D or r yields the other variable.
- Thus, one approach is to "guess" the equilibrium and iterate until we guess correctly.

A Solution Technique: The Shooting Algorithm

- Guess r. Yields K_D and w from $r = F_K(K_D, L)$ and $w = F_L$.
- Now, given this price, calculate the *individual* savings rule.
- Simulate the economy far enough into future to reach a steady-state distribution of capital.
- Check and see if $K_D = K_S$.
- If not, adjust guess of interest rate according to following:

$$r' = r + \lambda(K_D - K_S) \tag{11}$$

- where $\lambda < 1$

A Solution Technique: The Shooting Algorithm

- Adjusting interest rates:

$$r' = r + \lambda(K_D - K_S) \tag{12}$$

- If $K_S > K_D$: too much savings.
- Interest rate must fall to be in equilibrium.

First iteration

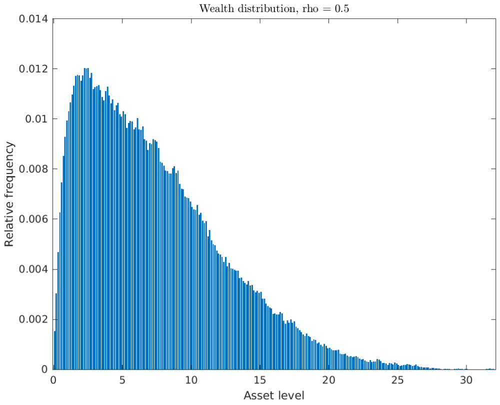
- Initial guess:
  - $r_0 = 0.03093$
- Three aggregates:
  1. $K = 8.8342$
  2. $L = 0.8582$
  3. $\rightarrow\ r = F_K = 0.0204$
- $r - r_0 < errtol$? 0.0309 – 0.0204 too large.
- Algorithm: fzero $\rightarrow$ pick local r_1 and try again.

Second iteration

- Initial guess:
  - $r_0 = 0.0308$
- Three aggregates:
  1. $K = 1.4531$
  2. $L = 0.9351$
  3. $\rightarrow\ r = F_K = 0.1985$
- $r - r_0 < errtol$? 0.0309 – 0.1935 too large.
- Very sensitive to r_0!

Converged Wealth Dist.

- Final wealth distribution after convergence:



Another Solution technique: Root-Finding and Excess Demand

- Functionally, this is the same as what we just did.
- Suppose we solve household decision rules k, and r.
- Then, the excess demand function is

$$\Delta(r) = K_D(r) - K_S(r) \tag{13}$$

- Where we have solved K_D for many values of r and have an expression for $K_S(r)$ (static firm optimization).
- Do one-dimensional root finding, i.e., find r^* such that

$$0 = \Delta(r^*) = K_D(r^*) - K_S(r^*) \tag{14}$$

Aggregate Uncertainty

- In a production economy, the agent's problem is given by

$$
\begin{aligned}
V(k,\epsilon;z,\psi) &= u(c) + \beta E[V(k',\epsilon';z',\psi')] && (15)\\
\text{s.t. } c + k' &\leq (1 + r(z,K,L) - \delta)k + w(z,K,L)\epsilon && (16)\\
k' &\geq \underline{k} && (17)\\
z' &= \text{ Markov} P(z'|z) && (18)\\
\epsilon &\sim \text{ Markov} P(\epsilon'|\epsilon, z') && (19)\\
\psi' &= \Psi(\psi, z, z') && (20)\\
c \geq 0, k \geq 0, k_0 &\text{ given}, z_0 \text{ given} && (21)
\end{aligned}
$$

- ϵ is a markov process for employment $\epsilon \in \{0, 1\}$
- Ψ is an unspecified evolution of the aggregate state.
- z *also* evolves as a markov process.
- Prices are determined from the firm's problem.

Prices - The Firm's Problem

- How we handle prices determines the difficulty of this problem.
- In this economy, a single firm produces using labor (hours) and capital.

$$\Pi = \max_{K,L} zF(K, L) - wL - rK \tag{22}$$

- This yields standard competitive prices for the rental rates.

Laws of Motion

- The future aggregate state enters the probability of employment.
- This means that it impacts **all** of the laws of motion:

$$z' = \text{Markov} P(z'|z) \tag{23}$$

$$\epsilon \sim \text{Markov} P(\epsilon'|\epsilon, z') \tag{24}$$

$$k' \leq (1 + r(z, K, L) - \delta)k + w(z, K, L)\epsilon - c \tag{25}$$

$$\psi' = \Psi(\psi, z, z') \tag{26}$$

- Because shocks to z change employment status and prices.

Recursive Competitive Equilibrium

- An RCE is given by pricing functions r, w, a worker value function $V(k, \epsilon, z; \psi)$, worker decision rules k', c, a type-distribution $\psi(k, \epsilon)$, and aggregates K and L that satisfy
    1. k' and c are the optimal solutions to the worker's problem given prices.
    2. Prices are formed competitively from the firm's problem.
    3. Consistency between aggregate evolution and individual decision rules: ψ is the distribution implied by worker decision rules given the aggregate state.
    4. Aggregates are consistent with individual policy rules: $K = \int k d\psi$, $L = \int \epsilon d\psi$

Type Distribution

- The type distribution is a problem.
- Each policy function and transition depends on the type distribution.
- But the type distribution is time-varying in response to aggregate shocks.
- Alternative: use a smaller number of moments that can be calculated quickly to characterize the type distribution.
- Like a "sufficient statistic" for the type distribution.

Krusell and Smith (1998)

- Specify moments from the type distribution γ that approximate the type distribution.
- Then: $\gamma' = \Gamma(\gamma, z, z')$.
- Household predicts prices using Γ instead of Ψ
- As long as this law of motion is reasonably accurate, this approximation will work.
- Krusell and Smith:
  - Pick first j moments of distribution over k, ϵ
  - i.e., mean, standard deviation,...
  - Use this as the law of motion.
- Use means: $ln(K') = \phi_0^z + \phi_1^z ln(K)$

Approximate problem

- In a production economy, the agent's problem is given by

$$V(k,\epsilon;z,K) = u(c) + \beta E[V(k',\epsilon';z',K')] \quad (27)$$
$$\text{s.t. } c + k' \leq (1 + r(z,K,L) - \delta)k + w(z,K,L)\epsilon \quad (28)$$
$$k' \geq \underline{k} \quad (29)$$
$$z' = \text{ Markov} P(z'|z) \quad (30)$$
$$\epsilon \sim \text{ Markov} P(\epsilon'|\epsilon, z') \quad (31)$$
$$ln(K') = \phi_0^z + \phi_1^z ln(K) \quad (32)$$
$$c \geq 0, k \geq 0, k_0 \text{ given}, z_0 \text{ given} \quad (33)$$

- LLN $\rightarrow$ N known given z.
- Now: need aggregate capital and ϕ_0^z, ϕ_1^z.
- Note: ϕ_0^z, ϕ_1^z *for each z*

KS Solution Technique

- Algorithm:
  1. Specify an initial forecasting function for K: $ln(K') = \phi_0^z + \phi_1^z ln(K)$. Pick initial values for ϕ_0^z, ϕ_1^z
  2. Tell household that the evolution of the aggregate state is given by $ln(K') = \phi_0^z + \phi_1^z ln(K)$. i.e., replace the previous constraint.
  3. Use value function iteration on this problem to solve for optimal policy rules.
  4. Simulate model forward to obtain K, z series. Drop first X number of observations.
  5. Use OLS on K, z series to see if forecasting was correct $|[\phi_0^z, \phi_1^z]' - \phi_0^{z'}, \phi_{1'}^z]| < errtol$
  6. If not, update ϕ_0^z, ϕ_1^z between initial and estimates.
- Another way to think about this: You estimated the slope and intercept of K' on some series $\{K_j, z_j\}_{j=1}^{j=t}$ and you are assessing its out of sample fit on $\{K_j, z_j\}_{j=t+1}^{T}$

KS Solution Technique

- Why does mean work?
- Linearity:

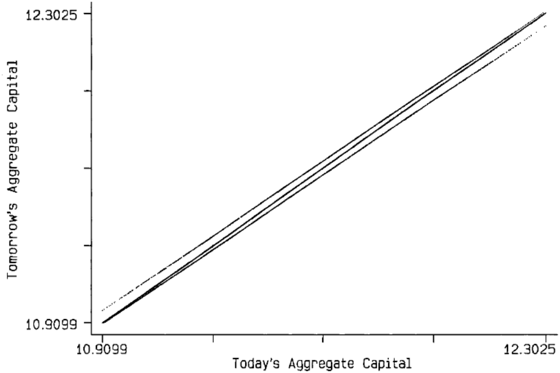


FIG. 1.—Tomorrow's vs. today's aggregate capital (benchmark model)

What do they find?

- With β heterogeneity, can hit wealth dist.

TABLE 1

DISTRIBUTION OF WEALTH: MODELS AND DATA

Model	Percentage of Wealth Held by Top 1%	5%	10%	20%	30%	Fraction with Wealth < 0	Gini Coefficient
Benchmark model	3	11	19	35	46	0	.25
Stochastic-β model	24	55	73	88	92	11	.82
Data	30	51	64	79	88	11	.79

- What is heterogeneity in β a reduced-form for?

Business Cycle Effects

- This model is built to handle stochastic shocks.
- How do heterogeneous agents respond over a business cycle?

TABLE 2

AGGREGATE TIME SERIES

Model	Mean(k_t)	Corr(c_t, y_t)	Standard Deviation (i_t)	Corr(y_t, y_{t-4})
Benchmark:				
Complete markets	11.54	.691	.031	.486
Incomplete markets	11.61	.701	.030	.481
$\sigma = 5$:				
Complete markets	11.55	.725	.034	.551
Incomplete markets	12.32	.741	.033	.524
Real business cycle:				
Complete markets	11.56	.639	.027	.342
Incomplete markets	11.58	.669	.027	.339
Stochastic-β:				
Incomplete markets	11.78	.825	.027	.459

Conclusion

- Today: solving heterogeneous agent models.
- Code to do this on the cluster.
- Next time (11/16): Huggett, Ventura, and Yaron (2011)
- Start your model projects!