Solving and Estimating Hansen's (1985) Model

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Motivation

- DSGE models are a standard tool used in macroeconomics.
    1. Historically, macroeconomists have used calibration as a relatively informal way to estimate parameters.
    2. Recently, economists have begun to employ more advanced statistical techniques like maximum likelihood.
    3. With knowledge of these more advanced techniques, we might be able to explore more important issues, like identification.
- Here: provide background for maximum likelihood estimation and calibration and compare the results.

Motivation

- Standard RBC: all fluctuations of hours worked on the *intensive* margin, i.e. average number of hours worked.
- Data: little fluctuation in average hours worked; lots of fluctuation in whether or not people are working (*extensive* margin).
- Standard RBC: missed badly on labor fluctuations (Frisch Elasticity, i.e. response of labor to change in wage too low).
- Solution: Modify model to have extensive margin with high Frisch Elasticity.
- Now: households pick the *probability* of working, but have to work a set number of hours.
- This is a *nonconvexity* in that it forces individuals to work either 0 or h hours.

Hansen (1985)

- Neoclassical growth model with labor-leisure lottery.
- A social planner maximize the following:

$$E(\sum_{t=0}^{\infty} \beta^t [ln(C_t) - \gamma H_t] \tag{1}$$

- Subject to the following constraints:

$$Y_t = A_t K_t^{\theta} (\eta^t H_t)^{1-\theta} \tag{2}$$

$$ln(A_t) = (1-\rho) ln(A) + \rho ln(A_{t-1}) + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2) \tag{3}$$

- The goods market clears and capital evolves in a predetermined fashion.
- Here, we assume that per capita labor productivity grows at rate η.

Equilibrium

- First step: detrend appropriate variables by per capita growth to get stationarity: i.e. $y_t = Y_t/\eta^t$.
- The system of equations that characterize the equilibrium are:

$$y_t = a_t k_t^{\theta} h_t^{1-\theta} \quad (4)$$

$$ln(a_t) = (1-\rho)ln(A) + \rho ln(a_{t-1}) + \epsilon_t \quad (5)$$

$$y_t = c_t + i_t \quad (6)$$

$$\eta k_{t+1} = (1-\delta)k_t + i_t \quad (7)$$

- Combine FOC[c] and FOC[h]:

$$\gamma c_t h_t = (1-\theta)y_t \quad (8)$$

- Euler Equation:

$$\frac{\eta}{c_t} = \beta E_t[\frac{1}{c_{t+1}}(\theta(\frac{y_{t+1}}{k_{t+1}}) + 1 - \delta)] \quad (9)$$

Solving for the Steady-State

$$
\begin{aligned}
ln(a^*) &= (1-\rho)ln(A) + \rho ln(a^*) \\
&\Rightarrow ln(a^*) = ln(A) \qquad (10)
\end{aligned}
$$

Euler Equation:

$$
\frac{\eta}{c^*} = \beta E_t[\frac{1}{c^*}(\theta(\frac{y^*}{k^*}) + 1 - \delta)]
$$

$$
\Rightarrow \frac{\eta}{\beta} = \theta\frac{y^*}{k^*} + 1 - \delta
$$

$$
\Rightarrow k^* = (\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})y^* \qquad (11)
$$

Solving for the Steady-State

- Use the previous to solve for investment

$$\eta k^* = (1-\delta)k^* + i^*$$

$$\Rightarrow (\eta - 1 + \delta)k^* = i^*$$

$$\Rightarrow i^* = (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})y^* \quad (12)$$

- FOC[c] and FOC[h]:

$$\gamma c^* h^* = (1-\theta)y^*$$

$$\Rightarrow \gamma[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]y^* h^* = (1-\theta)y^*$$

$$\Rightarrow h^* = (\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1} \quad (13)$$

Solving for the Steady-State

- Finally, solve for output.

$$y^* = a^* k^{*\theta} h^{*1-\theta}$$

$$y^* = a^*((\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})y^*)^\theta[(\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1}]^{1-\theta}$$

$$y^{*1-\theta} = a^*(\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})^\theta[(\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1}]^{1-\theta}$$

$$y^* = a^{*\frac{1}{1-\theta}}(\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})^{\frac{\theta}{1-\theta}}[(\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1}]^{1-\theta} \tag{14}$$

- All variables now a function of parameters.

Steady-States

- In steady-state $y_t = y_{t+1} = y^*$.

$$ln(a^*) = ln(A) \tag{15}$$

$$k^* = (\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})y^* \tag{16}$$

$$i^* = (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})y^* \tag{17}$$

$$c^* = [1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]y^* \tag{18}$$

$$h^* = (\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1} \tag{19}$$

$$y^* = a^{*\frac{1}{1-\theta}}(\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta})^{\frac{\theta}{1-\theta}}[(\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1}]^{1-\theta} \tag{20}$$

- These steady-states will be used for calibration/solving.

Overview

- Broadly, two methods of solving models:
  1. Local linear methods.
  2. Global non-linear methods.
- Tradeoff: accuracy (global non-linear) for speed and simplicity (local linear).
- My preference: global methods (linear methods involve linearizing Euler Equation, distorting choices over risk).
- Here: Discuss grid search/interpolation and Blanchard and Kahn's Method.

Grid search/ interpolation

- We have done this, so cover in brief for this particular model.
- Write down value function for problem.
- Loop over states for each time period (K, A).
- Maximize value for each choice (K', H).
- Note: contract specifies that the household gets paid *whether or not it works* so HH knows its budget with certainty once it picks H.
- Iterate to convergence.

Grid search/ interpolation

- Practically, loop over four things:
  1. Outermost loop: Capital.
  2. 2nd: Aggregate shock this period. This gives you the value function for each possible state.
  3. 3rd: Hours worked choices. This allows the household to pick its budget as a trade-off against the possibility of leisure.
  4. 4th: Capital next period. Sum over the possible shocks next period for each value of capital next period.
- Solve by picking the maximum in the 4th loop, conditional on each triple (K, A, H).
- Pick the maximum in the third loop, conditional on each (K, A) tuple. Note: for each H, you have already picked the maximum K'.
- Now, for each state (K, A), you have the value.
- Check error condition and iterate if convergence not achieved.

Local Linear Methods

- Log-linearize the system around the steady-state, then proceed.
- First have to solve the system for stability:
    1. Klein's Method (2000): Used for singular matrices.
    2. Sim's Method (2001): Used when it is unclear which variables are states and controls.
    3. Blanchard and Kahn's Method (1980): First solution method for rational expectations models.
- Here, we will use Blanchard and Kahn's Method.
- After solving the system, we apply the Kalman Filter until we find likelihood maximizing parameters.

Log-Linearizing the System

- Let $\tilde{y}_t = ln(y_t) - ln(y^*)$. Then, using Taylor Series approximations, the system characterizing the equilibrium becomes:

$$\tilde{y}_t = \tilde{a}_t + \theta \tilde{k}_t + (1-\theta)\tilde{h}_t \tag{21}$$

$$\tilde{a}_t = \rho \tilde{a}_{t-1} + \epsilon_t \tag{22}$$

$$(\frac{\eta}{\beta} - 1 + \delta)\tilde{y}_t = [\frac{\eta}{\beta} - 1 + \delta - \theta(\eta - 1 + \delta)]\tilde{c}_t + \theta(\eta - 1 + \delta)\tilde{i}_t \tag{23}$$

$$\eta \tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + (\eta - 1 + \delta)\tilde{i}_t \tag{24}$$

$$\tilde{y}_t = \tilde{c}_t + \tilde{h}_t \tag{25}$$

$$0 = \frac{\eta}{\beta}\tilde{c}_t + E[(\frac{\eta}{\beta} - 1 + \delta)(\tilde{y}_{t+1} - \tilde{k}_{t+1}) - \frac{\eta}{\beta}\tilde{c}_{t+1}] \tag{26}$$

Log-Linearizing the System

- We can now write the system as:

$$\Psi_1 \zeta_t = \Psi_2 \xi_t + \Psi_3 \tilde{a}_t \tag{27}$$

$$\Psi_4 E_t(\xi_{t+1}) = \Psi_5 \xi_t + \Psi_6 \zeta_t + \Psi_7 \tilde{a}_t \tag{28}$$

- ζ_t are static predetermined and nonpredetermined variables, $[\tilde{y}_t, \tilde{h}_t, \tilde{i}_t]'$.
- ξ_t are dynamic predetermined and nonpredetermined variables, $[\tilde{k}_t, \tilde{c}_t]'$.
- $\tilde{a}_t$ is the technology process.
- Why is $\tilde{c}_t$ among the dynamic variables?

Matrices

$$\kappa = \eta/\beta - 1 + \delta$$

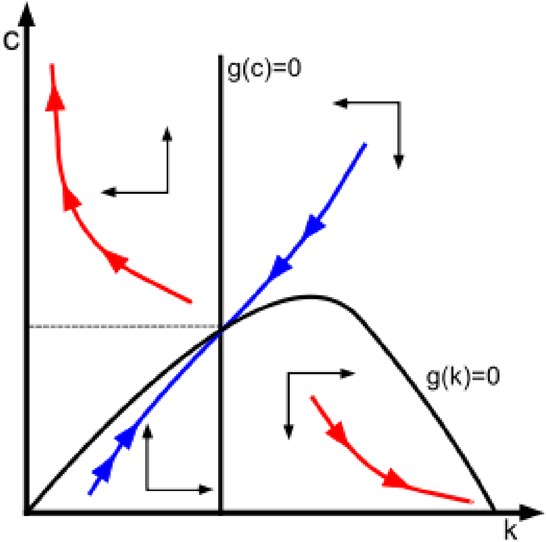
$$\lambda = \eta - 1 + \delta$$

$$\zeta_t = \begin{bmatrix} \tilde{y}_t & \tilde{i}_t & \tilde{h}_t \end{bmatrix}', \quad \tilde{\xi}_t = \begin{bmatrix} \tilde{k}_t & \tilde{c}_t \end{bmatrix}'$$

$$\Psi_1 = \begin{bmatrix} 1 & 0 & \theta - 1 \\ \kappa & -\theta\lambda & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} \theta & 0 \\ 0 & \kappa - \theta\lambda \\ 0 & 1 \end{bmatrix}, \quad \Psi_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Psi_4 = \begin{bmatrix} \eta & 0 \\ \kappa & \eta/\beta \end{bmatrix}, \quad \Psi_5 = \begin{bmatrix} 0 & 0 & 0 \\ -\kappa & 0 & 0 \end{bmatrix}, \quad \Psi_6 = \begin{bmatrix} 1 - \delta & 0 \\ 0 \eta/\beta & \end{bmatrix}, \quad \Psi_7 = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving the Model - Blanchard and Kahn (1980)



- Select $\tilde{c}_0$ st the system isn't explosive (optimal control!).

Solving the Model - Cont.

- Solve systems (29 - 30) so that ξ_{t+1} is only a function on ξ_t and $\tilde{a}_t$:

$$\Psi_1 \zeta_t = \Psi_2 \xi_t + \Psi_3 \tilde{a}_t \quad (29)$$

$$\Psi_4 E_t(\xi_{t+1}) = \Psi_5 \xi_t + \Psi_6 \zeta_t + \Psi_7 \tilde{a}_t \quad (30)$$

$$\Rightarrow \zeta_t = \Psi_1^{-1}[\Psi_2 \xi_t + \Psi_3 \tilde{a}_t]$$

- Plug into transition equation:

$$\Psi_4 E_t(\xi_{t+1}) = \Psi_5 \xi_t + \Psi_6 \Psi_1^{-1}[\Psi_2 \xi_t + \Psi_3 \tilde{a}_t] + \Psi_7 \tilde{a}_t$$

$$\Rightarrow E_t(\xi_{t+1}) = \Psi_4^{-1}[\Psi_5 + \Psi_6 \Psi_1^{-1} \Psi_2] \xi_t + \Psi_4^{-1}[\Psi_7 + \Psi_6 \Psi_1^{-1} \Psi_3] \tilde{a}_t \quad (31)$$

- Desired result!

Solving the Model - Cont.

- Having solved systems (29 - 30) so that ξ_{t+1} is only a function on ξ_t and $\tilde{a}_t$:

$$\begin{bmatrix} \tilde{k}_{t+1} \\ E_t(\tilde{c}_{t+1}) \end{bmatrix} = \Lambda^{-1} J \Lambda \begin{bmatrix} \tilde{k}_t \\ \tilde{c}_t \end{bmatrix} + E\tilde{a}_t \tag{32}$$

- $\Lambda^{-1} J \Lambda$ is the Jordan Decomposition.
- Subsume Λ into the model variables, denoted by hats:

$$\hat{c}_t = \Lambda_{12}\tilde{k}_t + \Lambda_{22}\tilde{c}_t \tag{33}$$

Solving the Model - Cont.

- Subsume Λ into the model variables, denoted by hats.

$$\begin{bmatrix} \hat{k}_{t+1} \\ E_t(\hat{c}_{t+1}) \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + D\tilde{a}_t \tag{34}$$

$$E_t(\hat{c}_{t+1}) = J_2\hat{c}_t + D_2\tilde{a}_t \tag{35}$$

- $J_2 > 1 \rightarrow$ bad choice of c_t and this explodes.
- Solution: pick c_t so that it isn't a function of c_{t-1}!
- Rearranging:

$$\hat{c}_t = J_2^{-1}E_t(\hat{c}_{t+1}) - J_2^{-1}D_2\tilde{a}_t \tag{36}$$

Solving the Model - Cont.

- Iterating on (33):

$$\hat{c}_{t+1} = J_2^{-1} E_t(\hat{c}_{t+2}) - J_2^{-1} D_2 \tilde{a}_{t+1} \quad (37)$$

$$\Rightarrow \hat{c}_t = J_2^{-1} E_t(J_2^{-1} E_t(\hat{c}_{t+2}) - J_2^{-1} D_2 \tilde{a}_{t+1}) - J_2^{-1} D_2 \tilde{a}_t$$

$$\Rightarrow \hat{c}_t = J_2^{-2} E_t(\hat{c}_{t+2})) - J_2^{-2} D_2 \rho \tilde{a}_t - J_2^{-1} D_2 \tilde{a}_t \quad (38)$$

- Impose transversality condition (i.e. $E_t(\hat{c}_{t+i})) = 0$ for large enough i):

$$\Rightarrow \hat{c}_t = -\sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 \rho \tilde{a}_t \quad (39)$$

Solving the Model - Cont.

- Iterating on (33):

$$\hat{c}_t = \Lambda_{12}\tilde{k}_t + \Lambda_{22}\tilde{c}_t$$

$$\Rightarrow \Lambda_{22}\tilde{c}_t = -\Lambda_{12}\tilde{k}_t - \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 \rho \tilde{a}_t$$

- Solving this yields:

$$\Rightarrow c_t = -\Lambda_{22}^{-1}\Lambda_{12}\tilde{k}_t + (1/\Lambda_{22})(\frac{D_2}{\rho - J_2})\tilde{a}_t \qquad (40)$$

- The system will now be saddle-path stable.

Formally

- Calibration is mathematically equivalent to just-identified GMM.
- Select a set of moments that we believe have a "high signal-to-noise" ratio.
- Generally, choose parameter so that steady-state variables match well-known quantities.

$$\Omega(\{X_t^M\}_{t=1}^T) = \Omega(\{X_t\}_{t=1}^T) \tag{41}$$

- Informally, use other implied moments to consider the "fit" of these parameters.

Selecting Moments for Hansen's Model

- We will start by considering a relationship between wages and output:

$$w_t = \frac{\partial y_t}{\partial h_t} = (1-\theta)a_t(\frac{k_t}{h_t})^{\theta} \tag{42}$$

$$\Rightarrow \frac{w_t h_t}{y_t} = (1-\theta) \tag{43}$$

- That is, our theory implies that the ratio of real wages to output should equal $1-\theta$, or the share of income paid to workers.

$$ln(Y_{t+1}) - ln(Y_t) \approx (1-\theta)ln(\eta) \tag{44}$$

- If we assume that the capital stock is approximately constant quarter to quarter, then this might be a reasonable approximation, given that A and H have little trend.

Selecting Moments for Hansen's Model - Cont.

- Cooley (1995) suggests that the steady-state capital-output ratio is 3.32 yearly:
- Then β^4 solves equation (11):

$$3.32 = \frac{\theta}{\frac{4\eta}{\beta} - 1 + 4\delta} \quad (45)$$

- We also take $\delta = 0.012$ from Cooley.
- Hours have been observed to be roughly trendless, thus we can find γ from the following:

$$h^* = (\frac{1-\theta}{\gamma})[1 - (\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta})]^{-1} \quad (46)$$

- From the following, we can estimate TFP and its associated parameters, ρ and σ_ϵ:

$$\Delta ln(Y_t) - [(1-\theta)[\Delta ln(H_t) + ln(\eta)] \approx \Delta ln(A_t) \quad (47)$$

Readying the Data

- We must match theoretical moments to the correct empirical moments:
    1. Our model doesn't include government or international trade, so these need to be removed from GDP.
    2. Use personal consumption and private investment.
    3. We have no prices, so each variable needs to be in real terms.
    4. Each of the variables is defined to be per-capita, so we need to divide by population.
- Further preparations are needed:
    1. Series decomposed into trend and cycle using Hodrick-Prescott Filter.
    2. Solving for θ requires further detrending: divide per-capita variable by η^t.
- Most of the data taken from BEA.
- Real wages per capita are taken from FRED, and not explicitly in the model.

Calibration Results

Table: Calibration Estimates

Preferences			Technology			
β	γ	θ	η	δ	ρ	σ_ϵ
0.9903	0.0076	0.3739	1.0061	0.0120	0.9972	0.0129

Table: Steady-States

y^*	c^*	i^*	h^*	k^*	a^*
8,834	6,694	2,140	108.61	118,320	17.8309

Kalman Filter

- With equation (40), we can now write the system in state-space form:

$$f_t = \Pi_1 s_t + \eta_t \tag{48}$$

$$s_{t+1} = \Pi_2 s_t + \epsilon_t \tag{49}$$

- We typically include η_t as measurement errors for the observed variables to avoid stochastic singularity.
- Having written the model like this, we can apply the Kalman Filter for different parameter values to find the likelihood maximizing parameter vector.

Comparing Results

Table: MLE Results Fixing β and δ

Preferences		Technology				
β	γ	θ	η	δ	ρ	σ_ϵ
0.99	0.0045	0.2292	1.0051	0.0250	0.9987	0.0052

Table: Calibration Estimates

Preferences		Technology				
β	γ	θ	η	δ	ρ	σ_ϵ
0.9903	0.0076	0.3739	1.0061	0.0120	0.9972	0.0129

Rios-Rull et al. (2012)

- Attempt to compare calibrated and Bayesian results.
- Estimate Hansen's model with investment shocks and different labor supply elasticities.
- Three different calibration approaches to identifying elasticity:
    1. Use long-run hours worked: elasticity around 2.
    2. Use lotteries (equivalent to what we have done here): elasticity of ∞.
    3. Use estimates from microeconomic studies: between 0.2 - 0.76.
- The models result in around the same results if identifying assumption 3 is used.
- They conclude that identification is more important than estimation technique.

Conclusion

- MLE and calibration provide estimates that are relatively similar in this context.
- Others have shown similar results for more complex models (Rios et al., 2012).
- Rather than estimation technique, we should focus on identification.
- Future Work:
    1. Use these techniques to further explore identification in DSGE models.
    2. Compare results in different models.