

Macro II

Professor Griffy

UAlbany

Spring 2021

Introduction

- ▶ So far: building tools to think about dynamic models.
- ▶ Now (and mostly rest of class):
 - ▶ Build on those tools to make more applicable to economics.
 - ▶ Use those tools to model the macroeconomy
- ▶ Today:
 - ▶ Introduce dynamic programming
- ▶ Homework due Wednesday.

Dynamic Programming

- ▶ Basic idea:
 - ▶ We can express macro models in a sequential form.
 - ▶ If we can write them *recursively*, we get access to more tools to solve them.
- ▶ We will start with a generic representation, give some important theorems, then discuss its use.

Sequential Problem

- ▶ We can broadly state most macro (and economics problems in general) as

$$\sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, x_{t+1})$$

$$\text{s.t. } x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \dots$$

$$x_0 \in X \text{ given}$$

- ▶ A solution tells us x_t at any time t .

Recursive Problem

- ▶ We want to write the sequential problem recursively

$$v(x) = \sup_{y \in \Gamma(x)} [r(x, y) + \beta v(y)], \forall x \in X.$$

- ▶ *We can also find solutions to this problem that solve the sequential problem.*
- ▶ We can make statements about the existence and uniqueness of those solutions.
- ▶ These statements are often easier when expressed this way.

Some definitions

- ▶ Metric space: a set S together with a metric (distance function), $\rho : S \times S \Rightarrow R$, such that for all $x, y, z \in S$:
 1. $\rho(x, y) \geq 0$, equality iff $x = y$
 2. $\rho(x, y) = \rho(y, x)$
 3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$
- ▶ Complete metric space: A metric space (S, ρ) is complete if every Cauchy sequence converge to an element in S .
- ▶ Cauchy sequence: a sequence $\{x_n\}_{n=0}^{\infty}$ for which $\rho(x_n, x_m) < \epsilon$, any $\epsilon > 0$ for $n, m \geq N_\epsilon$
- ▶ i.e., a sequence that gets closer and closer together (think of a model converging to equilibrium).

Contraction Mapping

- ▶ If (S, ρ) is a complete metric space and $T : S \Rightarrow S$ is a contraction mapping with modulus β , then
 1. T has exactly one fixed point v in S , and
 2. for any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$, $n = 0, 1, 2, \dots$

Blackwell's Sufficient Conditions

- ▶ Let $X \subseteq R^I$, and let $B(X)$ be a space of bounded functions $f : X \Rightarrow R$, with the sup norm. Let $T : B(X) \Rightarrow B(X)$ be an operator satisfying
 1. (monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$;
 2. (discounting) there exists some $\beta \in (0, 1)$ such that

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \text{ all } f \in B(X), a \geq 0, x \in X$$

Blackwell's Sufficient Applied

- ▶ Simple problem:

$$(Tv)(k) = \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v(y)\}$$

- ▶ Monotonicity: $f, g \in B(X)$ and $f(x) \geq g(x)$, for all $x \in X$, implies $(Tf)(x) \geq (Tg)(x)$, for all $x \in X$;
- ▶ define $g(x) \geq v(x)$, then

$$\begin{aligned}(Tg)(k) &= \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta g(y)\} \\ &\geq \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v(y)\} \\ &= (Tv)(k)\end{aligned}$$

- ▶ To see, take difference. $g(y) \geq v(y) \rightarrow$ monotone.

Blackwell's Sufficient Applied

- ▶ Simple problem:

$$(Tv)(k) = \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v(y)\}$$

- ▶ (discounting) there exists some $\beta \in (0, 1)$ such that

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \text{ all } f \in B(X), a \geq 0, x \in X$$

$$\begin{aligned}(Tv)(k + a) &= \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta[v(y) + a]\} \\ &= \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v(y) + \beta a\} \\ &= (Tv)(k) + \beta a\end{aligned}$$

- ▶ Thus, contraction mapping. Existence and uniqueness.

Theorem of the Maximum

- ▶ Broadly stated, the problem we face is

$$(Tv)(x) = \sup_y [F(x, y) + \beta v(y)]$$

s.t. y feasible given x

- ▶ This is just a value function
- ▶ With a specified constraint.

Correspondences

- ▶ We will define a correspondence $\Gamma(x)$ as
 - ▶ a set of feasible values of $y \in Y$ for $x \in X$,
 - ▶ where X can be thought of as the set of possible states
 - ▶ and Y the set of possible choices.
- ▶ The easiest example: the budget constraint.
- ▶ There are many feasible choices,
- ▶ we will pick on the maximizes the return function.
- ▶ Argmax correspondence:
 - ▶ We define a policy function $G(x)$ as a correspondence, where
 - ▶ $G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$

Compact Sets

- ▶ A compact set is a set that
 1. is closed: contains all of its limit points.
 2. is bounded: all points are within a finite distance of each other.
- ▶ Useful: most often applied to choice sets.
- ▶ Means that choices are finite and feasible.

Upper and Lower Hemi-Continuity

- ▶ Two notions of continuity, (really) loosely:
 1. Upper hemi-continuity: any choice y is in the set $\Gamma(x)$ (closed).
 2. Lower hemi-continuity: nearby x are in $\Gamma(x)$.

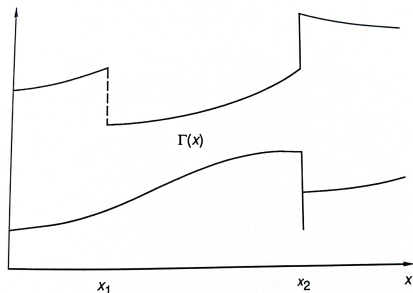


Figure 3.2

- ▶ Lower hemi-continuity: x_2 not lhc
- ▶ Upper hemi-continuity: x_1 not uhc

Upper and Lower Hemi-Continuity

- ▶ Upper hemi-continuity is useful:
 - ▶ Upper hemi-continuity preserves compactness:
 - ▶ if $C \subseteq X$ is compact and Γ is uhc,
 - ▶ $\Gamma(C)$ is compact.
- ▶ So if we place restrictions on X , our choice set is still in the correspondence.
- ▶ Allows our maximization problems to have solutions.
- ▶ If Γ is single-valued and uhc, it is continuous.

Theorem of the Maximum

- ▶ (conditions): Let $X \subseteq R^l$ and $Y \subseteq R^m$, let $f : X \times Y \Rightarrow R$ be a continuous function, and let $\Gamma : X \Rightarrow Y$ be a compact-valued and continuous correspondence.
- ▶ (implications): Then the function: $h : X \rightarrow R$ defined as $h(x) = \max_{y \in \Gamma(x)} f(x, y)$ and the correspondence $G : X \Rightarrow Y$ defined as $G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$ is
 1. nonempty,
 2. compact-valued, and
 3. upper hemi-continuous.
- ▶ Why is this useful?
 - ▶ under a few more assumptions (Γ is convex, f is strictly concave in y)
 - ▶ we can obtain the maximized value of f using the control g .
 - ▶ and as a result, $h(x)$.

Stochastic Dynamic Programming

Returning to our initial definition, let r be the return function and u the control vector with a state that evolves by $x_{t+1} = h(x_t, u_t, \varepsilon_{t+1})$. The sequential problem looks like

$$\begin{aligned} \max_{\{u_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \\ \text{s.t.} \quad x_{t+1} = h(x_t, u_t, \varepsilon_{t+1}) \quad \forall t, \quad x_0 \text{ given.} \end{aligned}$$

- ▶ where ε_t is some stochastic process (“shock”) with a defined support and some distribution function $F(\varepsilon)$
- ▶ we usually take this to be independent and identically distributed *or* Markov.

The Equilibrium

What is the equilibrium in this environment? What are the equilibrium objects?

- ▶ A sequence $\{u_t\}_{t=0}^{\infty}$ for every possible sequence of realizations for ε 's
- ▶ This is not so bad insofar as, at any given point in time, the problem has an infinite horizon and looks the same
- ▶ The above can be unwieldy, so we can instead find a *policy function* that tells the agent, at any point in time, what they should do given some observed x_t considering what they expect the ε 's to be in the future

The Recursive Problem

Now let's translate this into a recursive problem.

$$V(x) = \max_u \left\{ r(x, u) + \beta \mathbb{E} \left[V(\underbrace{h(x, u, \varepsilon')}_{x'}) \mid x \right] \right\}$$

where $\mathbb{E} \left[V(h(x, u, \varepsilon')) \mid x \right] \equiv \int_{\xi} V(h(x, u, \varepsilon')) dF(\varepsilon')$

How do we solve this? The obvious way: FOCs:

$$\frac{dV(x)}{du} = 0 : \quad r_2(x, u) + \beta \frac{d}{du} \mathbb{E} \left[V(h(x, u, \varepsilon')) \mid x \right] = 0$$

What allows us to pass the derivative through the expectation?

Differentiation under Integration

If the limits of integration *do not* depend on the control u , we can directly apply **Leibniz's rule** for differentiation under the integral (i.e., you just do it).

$$r_2(x, u) + \beta \mathbb{E} \left[\frac{dV(h(x, u, \varepsilon'))}{dx'} h_2(x, u, \varepsilon') \mid x \right] = 0$$

Alas, another roadblock: we do not know what $dV(x')/dx'$ is. Now we'll want to apply the Envelope Theorem. That is, we'll want to find $dV(x)/dx$.

Envelope Theorem

- ▶ The envelope theorem always seems to be a source of confusion.
- ▶ It states (loosely) that when we are maximizing a value function V with a choice x , we can proceed as though all other choices are at their optimal values.
- ▶ Why is this important? Because in principle, u affects the choice of u' .

$$r_2(x, u) + \beta \mathbb{E} \left[\frac{dV(h(x, u, \epsilon'))}{dx'} h_2(x, u, \epsilon') \mid x \right] = 0$$

$$r_2(x, u) + \beta \mathbb{E}[(r_1(x', u') + (r_2(x', u') + \beta \mathbb{E} \frac{\partial V}{\partial u'} h_2(x', u', \epsilon'')) \frac{\partial u'}{\partial x}) h_2(x, u, \epsilon') \mid x] = 0$$

$$r_2(x, u) + \beta \mathbb{E}[(r_1(x', u') + \cancel{(r_2(x', u') + \beta \mathbb{E} \frac{\partial V}{\partial u'} h_2(x', u', \epsilon'')) \frac{\partial u'}{\partial x}}) h_2(x, u, \epsilon') \mid x] = 0$$

- ▶ We can cancel future terms because we optimally pick u'
- ▶ i.e., we plug in $g(x)$ for u .

Envelope Theorem II

If the problem we are working with can be written in such a way such that the transition does not depend on x , this can be greatly simplified to

$$\frac{dV(x)}{dx} = r_1(x, u) \quad \implies \quad \frac{dV(x')}{dx'} = r_1(x', u').$$

Plugging this back into the FOC gives the stochastic EE.

$$r_2(x, u) + \beta \mathbb{E} [r_1(x', u') h_2(x, u, \varepsilon') | x] = 0$$

Now: return to neoclassical growth. Suppose that capital evolves according to $k' = (1 - \delta)k + a + \varepsilon$ (where ε is iid), and that there is full depreciation ($\delta = 1$).

Stochastic Neoclassical Growth

$$V(k, \varepsilon) = \max_{c, k'} \{ \ln(c) + \beta \mathbb{E} [V(k', \varepsilon')] \} \quad \text{s.t.} \quad c = k^\alpha - k' + \varepsilon$$

$$\implies V(k, \varepsilon) = \max_{k'} \{ \ln(k^\alpha - k' + \varepsilon) + \beta \mathbb{E} [V(k', \varepsilon')] \}$$

The FOC is given by

$$\frac{1}{k^\alpha - k' + \varepsilon} = \beta \mathbb{E} \left[\frac{dV(k', \varepsilon')}{dk'} \right],$$

where we passed the derivative through the integral using Leibniz's rule.

Solving

Now for the Envelope Theorem.

$$\frac{dV(k, \varepsilon)}{dk} = \frac{\alpha k^{\alpha-1}}{k^\alpha - k' + \varepsilon} \quad \Longrightarrow \quad \frac{dV(k', \varepsilon')}{dk'} = \frac{\alpha k'^{\alpha-1}}{k'^\alpha - k'' + \varepsilon'}$$

Plugging this back into the FOC, we have the EE (which we can rewrite however we want).

$$\frac{1}{k^\alpha - k' + \varepsilon} = \beta \mathbb{E} \left[\frac{\alpha k'^{\alpha-1}}{k'^\alpha - k'' + \varepsilon'} \right]$$

$$\frac{1}{c} = \beta \mathbb{E} \left[\frac{\alpha k'^{\alpha-1}}{c'} \right]$$

Next Time

- ▶ Next: Permanent Income and Consumption Smoothing
- ▶ Homework due on Wednesday.
- ▶ Note: The last part of HW3 is meant to be tricky.
- ▶ Try it, but if you can't get it that is okay.