# Macro II

Professor Griffy

UAlbany

Spring 2023

# Announcements

- No class Thursday (out of town for a conference)
- Office hours today: 1:30-2:30 (John Jones visiting the department)
- Today: Solving heterogeneous agent models.
- Idea:
  - Solving these models is non-trivial.
  - Must consider the state of every agent in economy.
- Homework on solving heterogeneous agent models.
- Start from the code on the cluster.

# Heterogeneous Agent Production Economy

- In a production economy, the agent's problem is given by

$$V(k,\epsilon;\psi) = u(c) + \beta E[V(k'\epsilon';\psi')] \quad (1)$$
$$\text{s.t. } c + k' \leq (1 + r(K,L) - \delta)k + w(K,L)\epsilon \quad (2)$$
$$k' \geq \underline{k} \quad (3)$$
$$\epsilon \sim \text{ Markov } P(\epsilon'|\epsilon) \quad (4)$$
$$\psi' = \Psi(\psi) \quad (5)$$
$$c \geq 0, k \geq 0, k_0 \text{ given} \quad (6)$$

- $\epsilon$ is a markov process that yields hours worked.
- $\Psi$ is an unspecified evolution of the aggregate state $(k,\epsilon)$.
- Prices are determined from the firm's problem

# Prices - The Firm's Problem

- How we handle prices determines the difficulty of this problem.
- In this economy, a single firm produces using labor (hours) and capital.

$$\Pi = \max_{K,L} F(K, L) - wL - rK \tag{7}$$

- This yields standard competitive prices for the rental rates.

# Stationary Recursive Competitive Equilibrium

- A stationary RCE is given by pricing functions $r, w$, a worker value function $V(k, \epsilon; \psi)$, worker decision rules $k', c$, a type-distribution $\psi(k, \epsilon)$, and aggregates $K$ and $L$ that satisfy
  1. $k'$ and $c$ are the optimal solutions to the worker's problem given prices.
  2. Prices are formed competitively from the firm's problem.
  3. Consistency between aggregate evolution and individual decision rules: $\psi$ is the stationary distribution implied by worker decision rules.
  4. Aggregates are consistent with individual policy rules: $K = \int k d\psi$, $L = \int \epsilon d\psi$

# Calibration

- Functions:
  - Utility: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$
  - Production: $F(K, L) = K^{\alpha} L^{1-\alpha}$
- Borrowing constraint: $\underline{k} = 0$
- $\alpha = 0.36$.

# Solving the Model: Market Clearing

- In equilibrium

$$K = \sum_k \sum_\epsilon k_s(k, \epsilon)\psi(k, \epsilon) \tag{8}$$

- where $k_s$ is the supply of savings.
- What must the equilibrium prices satisfy?

$$r = F_K(K_D, L) \tag{9}$$

$$K_D(r) = K_S(r) \tag{10}$$

- Fixing $K_D$ or $r$ yields the other variable.
- Thus, one approach is to "guess" the equilibrium and iterate until we guess correctly.

# A Solution Technique: The Shooting Algorithm

- Guess $r$. Yields $K_D$ and $w$ from $r = F_K(K_D, L)$ and $w = F_L$.
- Now, given this price, calculate the *individual* savings rule.
- Simulate the economy far enough into future to reach a steady-state distribution of capital.
- Check and see if $K_D = K_S$.
- If not, adjust guess of interest rate according to following:

$$r' = r + \lambda(K_D - K_S) \tag{11}$$

- where $\lambda < 1$

# A Solution Technique: The Shooting Algorithm

- Adjusting interest rates:

$$r' = r + \lambda(K_D - K_S) \tag{12}$$

- If $K_S > K_D$: too much savings.
- Interest rate must fall to be in equilibrium.

# First iteration

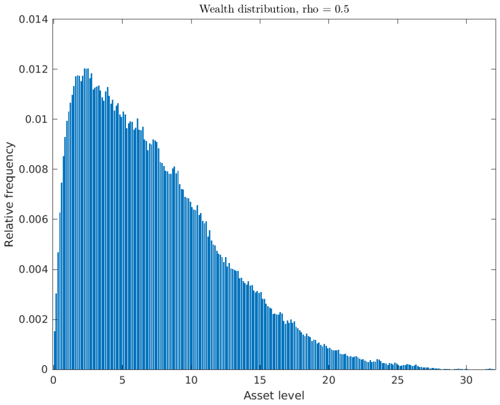
- Initial guess:
  - $r_0 = 0.03093$
- Three aggregates:
  1. $K = 8.8342$
  2. $L = 0.8582$
  3. $\rightarrow\ r = F_K = 0.0204$
- $r - r_0 < errtol$? 0.0309 – 0.0204 too large.
- Algorithm: fzero $\rightarrow$ pick local $r_1$ and try again.

# Second iteration

- Initial guess:
  - $r_0 = 0.0308$
- Three aggregates:
  1. $K = 1.4531$
  2. $L = 0.9351$
  3. $\rightarrow \; r = F_K = 0.1985$
- $r - r_0 < errtol$? 0.0309 – 0.1935 too large.
- Very sensitive to $r_0$!

# Converged Wealth Dist.

- Final wealth distribution after convergence:



# Another Solution technique: Root-Finding and Excess Demand

- Functionally, this is the same as what we just did.
- Suppose we solve household decision rules $k$, and $r$.
- Then, the excess demand function is

$$\Delta(r) = K_D(r) - K_S(r) \tag{13}$$

- Where we have solved $K_D$ for many values of $r$ and have an expression for $K_S(r)$ (static firm optimization).
- Do one-dimensional root finding, i.e., find $r^*$ such that

$$0 = \Delta(r^*) = K_D(r^*) - K_S(r^*) \tag{14}$$

# Aggregate Uncertainty

- In a production economy, the agent's problem is given by

$$
\begin{aligned}
V(k,\epsilon;z,\psi) &= u(c) + \beta E[V(k',\epsilon';z',\psi')] && (15)\\
\text{s.t. } c + k' &\leq (1 + r(z,K,L) - \delta)k + w(z,K,L)\epsilon && (16)\\
k' &\geq \underline{k} && (17)\\
z' &= \text{ Markov} P(z'|z) && (18)\\
\epsilon &\sim \text{ Markov} P(\epsilon'|\epsilon,z') && (19)\\
\psi' &= \Psi(\psi,z,z') && (20)\\
c \geq 0, k \geq 0, k_0 &\text{ given}, z_0 \text{ given} && (21)
\end{aligned}
$$

- $\epsilon$ is a markov process for employment $\epsilon \in \{0,1\}$
- $\Psi$ is an unspecified evolution of the aggregate state.
- $z$ *also* evolves as a markov process.
- Prices are determined from the firm's problem.

# Prices - The Firm's Problem

- How we handle prices determines the difficulty of this problem.
- In this economy, a single firm produces using labor (hours) and capital.

$$\Pi = \max_{K,L} zF(K, L) - wL - rK \tag{22}$$

- This yields standard competitive prices for the rental rates.

# Laws of Motion

- The future aggregate state enters the probability of employment.
- This means that it impacts **all** of the laws of motion:

$$z' = \text{Markov}P(z'|z) \tag{23}$$

$$\epsilon \sim \text{Markov}P(\epsilon'|\epsilon, z') \tag{24}$$

$$k' \leq (1 + r(z, K, L) - \delta)k + w(z, K, L)\epsilon - c \tag{25}$$

$$\psi' = \Psi(\psi, z, z') \tag{26}$$

- Because shocks to $z$ change employment status and prices.

# Recursive Competitive Equilibrium

- An RCE is given by pricing functions $r, w$, a worker value function $V(k, \epsilon, z; \psi)$, worker decision rules $k', c$, a type-distribution $\psi(k, \epsilon)$, and aggregates $K$ and $L$ that satisfy
  1. $k'$ and $c$ are the optimal solutions to the worker's problem given prices.
  2. Prices are formed competitively from the firm's problem.
  3. Consistency between aggregate evolution and individual decision rules: $\psi$ is the distribution implied by worker decision rules given the aggregate state.
  4. Aggregates are consistent with individual policy rules: $K = \int k d\psi$, $L = \int \epsilon d\psi$

# Type Distribution

- The type distribution is a problem.
- Each policy function and transition depends on the type distribution.
- But the type distribution is time-varying in response to aggregate shocks.
- Alternative: use a smaller number of moments that can be calculated quickly to characterize the type distribution.
- Like a “sufficient statistic” for the type distribution.

# Krusell and Smith (1998)

- Specify moments from the type distribution $\gamma$ that approximate the type distribution.
- Then: $\gamma' = \Gamma(\gamma, z, z')$.
- Household predicts prices using $\Gamma$ instead of $\Psi$
- As long as this law of motion is reasonably accurate, this approximation will work.
- Krusell and Smith:
  - Pick first $j$ moments of distribution over $k, \epsilon$
  - i.e., mean, standard deviation,...
  - Use this as the law of motion.
- Use means: $ln(K') = \phi_0^z + \phi_1^z ln(K)$

# Approximate problem

- In a production economy, the agent's problem is given by

$$
\begin{aligned}
V(k,\epsilon;z,K) &= u(c)+\beta E[V(k',\epsilon';z',K')] && (27)\\
\text{s.t. } c+k' &\leq (1+r(z,K,L)-\delta)k + w(z,K,L)\epsilon && (28)\\
k' &\geq \underline{k} && (29)\\
z' &= \text{ Markov} P(z'|z) && (30)\\
\epsilon &\sim \text{ Markov} P(\epsilon'|\epsilon,z') && (31)\\
ln(K') &= \phi_0^z + \phi_1^z ln(K) && (32)\\
c \geq 0, k &\geq 0, k_0 \text{ given}, z_0 \text{ given} && (33)
\end{aligned}
$$

- LLN $\rightarrow$ N known given $z$.
- Now: need aggregate capital and $\phi_0^z$, $\phi_1^z$.
- Note: $\phi_0^z$, $\phi_1^z$ *for each z*

# KS Solution Technique

- Algorithm:
  1. Specify an initial forecasting function for $K$: $ln(K') = \phi_0^z + \phi_1^z ln(K)$. Pick initial values for $\phi_0^z, \phi_1^z$
  2. Tell household that the evolution of the aggregate state is given by $ln(K') = \phi_0^z + \phi_1^z ln(K)$. i.e., replace the previous constraint.
  3. Use value function iteration on this problem to solve for optimal policy rules.
  4. Simulate model forward to obtain $K, z$ series. Drop first X number of observations.
  5. Use OLS on $K, z$ series to see if forecasting was correct $|[\phi_0^z, \phi_1^z]' - \phi_0^{z'}, \phi_{1'}^z]| < errtol$
  6. If not, update $\phi_0^z$, $\phi_1^z$ between initial and estimates.
- Another way to think about this: You estimated the slope and intercept of $K'$ on some series $\{K_j, z_j\}_{j=1}^{j=t}$ and you are assessing its out of sample fit on $\{K_j, z_j\}_{j=t+1}^{T}$

# KS Solution Technique

- Why does mean work?
- Linearity:

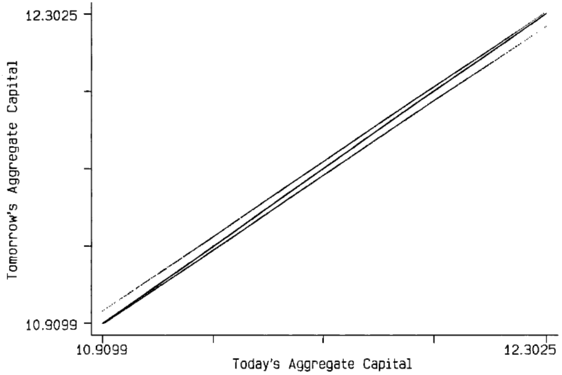



Fig. 1.—Tomorrow's vs. today's aggregate capital (benchmark model)

# What do they find?

- With $\beta$ heterogeneity, can hit wealth dist.

TABLE 1

DISTRIBUTION OF WEALTH: MODELS AND DATA

| Model | Percentage of Wealth Held by Top 1% | 5% | 10% | 20% | 30% | Fraction with Wealth < 0 | Gini Coefficient |
|---|---|---|---|---|---|---|---|
| Benchmark model | 3 | 11 | 19 | 35 | 46 | 0 | .25 |
| Stochastic-β model | 24 | 55 | 73 | 88 | 92 | 11 | .82 |
| Data | 30 | 51 | 64 | 79 | 88 | 11 | .79 |

- What is heterogeneity in $\beta$ a reduced-form for?

# Business Cycle Effects

- This model is built to handle stochastic shocks.
- How do heterogeneous agents respond over a business cycle?

TABLE 2

AGGREGATE TIME SERIES

| Model | Mean($k_t$) | Corr($c_t$, $y_t$) | Standard Deviation ($i_t$) | Corr($y_t$, $y_{t-4}$) |
|---|---|---|---|---|
| Benchmark: | | | | |
| Complete markets | 11.54 | .691 | .031 | .486 |
| Incomplete markets | 11.61 | .701 | .030 | .481 |
| $\sigma = 5$: | | | | |
| Complete markets | 11.55 | .725 | .034 | .551 |
| Incomplete markets | 12.32 | .741 | .033 | .524 |
| Real business cycle: | | | | |
| Complete markets | 11.56 | .639 | .027 | .342 |
| Incomplete markets | 11.58 | .669 | .027 | .339 |
| Stochastic-$\beta$: | | | | |
| Incomplete markets | 11.78 | .825 | .027 | .459 |

# Conclusion

- Next class next Tuesday.
- Today: solving heterogeneous agent models.
- Code to do this on the cluster.
- Start labor market frictions.