AECO 701 Midterm

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A Cake Eating Problem. Dwayne "The Rock" Johnson is starring in a reality TV series in which he is dropped on a remote island with a limited amount of food and no way to extract food from the island. Although he is in no real risk, Dwayne is an astute scholar of his craft and studies dynamic programming to better understand the decisions of someone abandoned on a deserted island. Dwayne writes down the following "cake eating" problem for the man on the deserted island:

$$\sum_{t=0}^{\infty} \beta^t ln(c_t) \tag{1}$$

Where $\beta \in (0, 1)$. The resource constraint is as follows:

$$c_t + k_{t+1} \leq k_t \tag{2}$$

where $c_t, k_{t+1} \geq 0$, and $k_0 > 0$ is given. Here, k_t is the amount of cake remaining. Please use this problem as the set-up for the following two questions.

1 [50] **Cake Eating and Contraction Mapping Theorem.** Use the cake-eating problem above to answer the following question:

a [10] Write the problem recursively and define the Equilibrium in The Rock's economy.

Answer:
We can define the worker's problem as the following:

$$V(k) = \max_{c,k'} ln(c) + \beta V(k') \tag{3}$$

$$\text{s.t. } c + k' = k \tag{4}$$

Equilibrium

An equilibrium in this model is a value function $V(k)$ and policy rules $c = g_c(k), k' = g_k(k)$ such that

(a) The policy functions c and k' solve the worker's problem.

(b) Capital evolves according to $k' = k - c$.

b [5] Solve for the Euler Equation.
Answer:
Do the standard steps. find FOC[c], FOC[k']:

$$FOC[c] : \frac{1}{c} - \lambda = 0 \tag{5}$$

$$FOC[k'] : -\lambda + \beta \frac{\partial V}{\partial k'} = 0 \tag{6}$$

Now take the derivative with respect to k and use the Envelope Theorem:

$$\frac{\partial V}{\partial k} = \lambda \tag{7}$$
$$\rightarrow \frac{\partial V}{\partial k'} = \lambda \tag{8}$$

Note that even though this is a cake eating problem and the marginal utility of consumption will clearly decrease over time, $V(k)$ is a contraction mapping, so λ will not vary with time. Now, combining all three yields the Euler Equation:

$$\frac{1}{c} = \beta\frac{1}{c'} \tag{9}$$

c [20] Guess that the value function takes the form $V(k) = a_0 + a_1 ln(k)$.

- Write out Blackwell's Sufficient Conditions for a contraction. There are two conditions: discounting and monotonicity.

  Answer:

  Theorem: Blackwell's Sufficient Conditions

  T is monotone if for $f(x) \leq g(x)\ \forall x \in X$, then

  $$Tf(x) \leq Tg(x) \quad \forall x \in X \tag{10}$$

  T discounts if for some $\beta \in (0,1)$ and any $a \in \mathcal{R}_+$

  $$T(f+a)(x) \leq Tf(x) + \beta a \quad \forall x \in X \tag{11}$$

- Use Blackwell's Sufficient Conditions to show that this guess is a contraction.

  Answer:

  This is just a straightfoward application of Blackwell's Sufficient Conditions. A general proof is given below

  Monotonicity:

  Let $f(x) \leq g(x)$. Then

  $$Tf(x) = h(x,y) + \beta f(x)$$
  $$Tg(x) = h(x,y) + \beta g(x)$$

  Taking the difference of these two yields:

  $$\begin{aligned} Tf(x) - Tg(x) &= h(x,y) + \beta f(x) - [h(x,y) + \beta g(x)] \\ &= \beta(f(x) - g(x)) \end{aligned}$$

  Since $f(x) \leq g(x)$, we know that $Tf(x) \leq Tg(x)$. Thus, the Bellman Operator is monotonic. In our context, $f(x) = a_0 + a_1 ln(x)$ and $g(x) = a_0 + a_1 ln(x)$ and this holds with equality.

  Discounting:

  Let f and h be functions. Then

  $$T(f+a)(x) \leq Tf(x) + \beta a$$

$$T(f+a)(x) = h(x,y) + \beta(f(x)+a)$$

$$\Rightarrow Tf(x) + \beta a = h(x,y) + \beta f(x) + \beta a$$

These are equivalent, so we see that β discounts. In our context, This is equivalent to showing that $f(x) = a_0 + a_1 ln(x)$ remains no larger than $g(x) + b = a_0 + b + a_1 ln(x)$, where $b > 0$, after applying the discount factor β to the problem.

d [15] Return to the initial set-up for the problem (i.e., no need to use your guess from the previous part). Now suppose that there is a probability δ that during any period there will be a drop of exactly $\hat{y} > 0$ units of food, cooked by The Rock. The agent smells the food that is being cooked by the Rock *between periods and it is known before the agent makes decisions* (italicized to emphasize that the should be known at the start of the period). Rewrite the recursive problem and find the Euler Equation. Assuming that k_0 is the same, is consumption higher, lower, or the same in this economy, and why?

Answer:

Now, there is a probability δ that The Rock will deliver food. There is some ambiguity here, depending on how you treat $\hat{y}$. If you believe it is non-storable (i.e., it will go bad), then agents consume all of it that period. If you interpret it as equivalent to more capital, they may not consume all of it. The problems are equivalent as long as $c^*(k+\hat{y}) \geq c^*(k) + \hat{y}$ where c^* is the policy function from part b. I'll write out both versions. First, if you see $\hat{y}$ as capital:

$$V(k,y) = \max_{c,k'} ln(c) + \beta[\delta V(k',\hat{y}) + (1-\delta)V(k',0)] \tag{12}$$

$$\text{s.t. } c + k' = k + y \tag{13}$$

If $\hat{y}$ is perishable or non-storable:

$$V(k,y) = \max_{c,k'} ln(c) + \beta[\delta V(k',\hat{y}) + (1-\delta)V(k',0)] \tag{14}$$

$$\text{s.t. } c = k + y - i \tag{15}$$

$$i = k' - k, \ i \leq 0 \tag{16}$$

Here, the two worlds are the same as long as $i < 0$. If $i = 0$, agents would like to save some of The Rock's cooking for the future. If $i = 0$, ie there is no interior solution, consumption is necessarily higher. Assuming an interior solution $i < 0$, the Euler Equation includes a gamble over two possible consumption states:

$$\frac{1}{c} = \beta[\delta \frac{1}{c'(k',\hat{y})} + (1-\delta)\frac{1}{c'(k',0)}] \tag{17}$$

You can easily appeal to Jensen's Inequality and note that more resources implies higher average c, which implies lower marginal utility, and hence more consumption today. The easier way to note that this is a closed economy with no positive investment, which means we can write the present value of lifetime resources as

$$\hat{K} = k_0 + \sum_{t=0}^{\infty} \delta\hat{y} > k_0 \tag{18}$$

Because utility is strictly increasing in c,

$$\hat{K} = \hat{C} = k_0 + \sum_{t=0}^{\infty} \delta \hat{y} > k_0 = \hat{C}_{No\ Rock} \tag{19}$$

where $\hat{C}$ is lifetime consumption and is clearly larger than the absence of the extra resources.

2 [50] **Guess and Verify.** Use the cake-eating problem above to answer the following question:

a [30] Guess that the value function takes the form $V(k) = a_0 + a_1 ln(k)$. Solve for a_0 and a_1, find the optimal policy functions for k' and c, and then verify your guess.

Answer:

Guessing that the value function takes the form $V(k) = a_0 + a_1 ln(k)$ and plugging this into the recursive problem yields

$$RHS = \max_{k'} ln(k - k') + \beta[a_0 + a_1 ln(k')]$$

Taking the derivative of this with respect to k' yields

$$\begin{aligned}
FOC[k'] &= -\frac{1}{k - k'} + \beta a_1 \frac{1}{k'} = 0 \\
\frac{1}{k - k'} &= \beta a_1 \frac{1}{k'} \\
k' &= \beta a_1 (k - k') \\
k' + \beta a_1 k' &= \beta a_1 k \\
(1 + \beta a_1) k' &= \beta a_1 k \\
k' &= \frac{\beta a_1 k}{1 + \beta a_1}
\end{aligned}$$

The decision rule for c is straightforward:

$$\begin{aligned}
c &= k - \frac{\beta a_1 k}{1 + \beta a_1} \\
&= (1 - \frac{\beta a_1}{1 + \beta a_1}) k \\
&= \frac{1}{1 + \beta a_1} k
\end{aligned}$$

Now, we can plug in these decision rules to solve for a_0 and a_1:

$$\begin{aligned}
RHS^* &= ln(\frac{1}{1 + \beta a_1} k) + \beta[a_0 + a_1 ln(\frac{\beta a_1 k}{1 + \beta a_1})] \\
&= ln(k) - ln(1 + \beta a_1) + \beta[a_0 + a_1\{ln(k) + ln(\beta a_1) - ln(1 + \beta a_1)\}] \\
&= -(1 + \beta a_1) ln(1 + \beta a_1) + \beta[a_0 + a_1 ln(\beta a_1)] + (1 + \beta a_1) ln(k)
\end{aligned}$$

We know that $a_1 = (1 + \beta a_1)$ and solving this gives us a_1:

$$\begin{aligned} a_1 &= (1 + \beta a_1) \\ (1 - \beta) a_1 &= 1 \\ a_1 &= \frac{1}{1 - \beta} \end{aligned}$$

Now plugging this in for a_0 yields

$$\begin{aligned} a_0 &= -(1 + \beta a_1) ln(1 + \beta a_1) + \beta[a_0 + a_1 ln(\beta a_1)] \\ (1 - \beta) a_0 &= -(\frac{1}{1 - \beta}) ln(1 + \frac{\beta}{1 - \beta}) + \frac{\beta}{1 - \beta} ln(\frac{\beta}{1 - \beta}) \\ a_0 &= \frac{-ln(1 + \frac{\beta}{1-\beta}) + \beta ln(\frac{\beta}{1-\beta})}{(1 - \beta)^2} \end{aligned}$$

Because this can be separated into an expression that follows $V(k) = a_0 + a_1 ln(k)$, our guess is verified.

b [20] Explain why guess and verify guesses generically take a form that looks like $V(x) = b_0 + b_1 u(x)$, where the utility function takes a hyperbolic form (the ones you've seen before, quadratic, exponential, power) and x is a (possibly a vector of) state variables. Explain means provide *intuition*.
Answer:
The easiest way to get intuition is to start by considering the following. Each agent maximizes lifetime utility:

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{20}$$

Let's call this $V_0(k)$, where the subscript refers to the current period. Now what is the relationship between $V_0(k)$ and $V_1(k)$?

$$V_0(k) = \max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{21}$$

$$V_1(k) = \max_{c_t} \sum_{t=1}^{\infty} \beta^t u(c_t) \tag{22}$$

$$V_0(k) = u(c_0) + \beta V_1(k) \tag{23}$$

Now, in a stationary infinite horizon model, neither the value function nor the policy functions can depend on time:

$$\rightarrow V_0(k) = V_1(k) = V(k) \tag{24}$$

Now plug this into our previous expression:

$$V_0(k) = u(c_0) + \beta V_1(k) \tag{25}$$

$$V(k) = u(c) + \beta V(k) \tag{26}$$

$$(1-\beta)V(k) = u(c) \tag{27}$$

$$V(k) = \frac{1}{(1-\beta)}u(c) \tag{28}$$

$$V(k) = \frac{1}{(1-\beta)}u(c(k)) \tag{29}$$

You can see that this is the a_1 coefficient and that consumption is *only* a function of k. And, we saw from earlier that $c = (1-\beta)k$, which shouldn't be a surprise since log utility typically results in fixed expenditure shares. What differs here is that workers cannot acquire more capital. What a_0 is doing intuitively is accounting for the slowly decreasing budget over time. Because preferences are homothetic, this shift in budgets does not affect choices (hence, a_1 is simple), but it does affect utility. a_0 is the adjustment in lifetime utility that precisely offsets the present discounted effects of an agents budget shifting downward over their lifetime.