

Macro II: Stochastic Processes I

Professor Griffy

UAlbany

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Introduction

- ▶ Today: start talking about time series/stochastic processes.
- ▶ Homework due tonight?
- ▶ Continue stochastic processes on Tuesday.

Stochastic Processes

- ▶ Random variables
- ▶ Conditional distributions
- ▶ Markov processes

Preliminaries

- ▶ X is a random variable, x is its realization
- ▶ Support: smallest set S such that $\Pr(x \in S) = 1$
- ▶ Cumulative distribution function: $F(x) = \Pr(X \leq x)$
- ▶ Density function: $f(x) = \frac{d}{dx} F(x)$ implying that $f(x) dx = dF(x)$

The Expected Value

- ▶ Mean is the expectation

$$\bar{X} = E(X) = \int_{-\infty}^{\infty} x dF(x) = \int_{-\infty}^{\infty} x f(x) dx$$

- ▶ The expectation of a function of a random variable, $g(X)$, is

$$E(g(X)) = \int_{-\infty}^{\infty} g(X) dF(x)$$

- ▶ Note that $E(g(X)) \neq g(\bar{X})$ unless $g(X)$ is linear, i.e.

$$g(X) = b \cdot X$$

The Variance

- ▶ Variance

$$V(X) = E[(X - \bar{X})^2]$$

- ▶ Standard deviation

$$[V(X)]^{\frac{1}{2}}$$

Jointly Distributed Random Variables

- ▶ Random vector (X, Y)
- ▶ Joint distribution function: $F(x, y) = \Pr(X \leq x, Y \leq y)$
- ▶ Covariance: $C(X, Y) = E[(X - \bar{X}) \cdot (Y - \bar{Y})]$
- ▶ Cross-correlation $= \frac{C(X, Y)}{[V(X) \cdot V(Y)]^{\frac{1}{2}}}$
- ▶ Expectation of a linear combination

$$E(aX + bY) = aE(X) + bE(Y)$$

What is a Stochastic Process?

- ▶ Stochastic process is an infinite sequence of random variables $\{X_t\}_{t=-\infty}^{\infty}$
- ▶ j'th autocovariance = $\gamma_j = C(X_t, X_{t-j})$
- ▶ Strict stationarity: distribution of $(X_t, X_{t+j_1}, X_{t+j_2}, \dots, X_{t+j_n},)$ does not depend on t
- ▶ Covariance stationarity: \bar{X}_t and $C(X_t, X_{t-j})$ do not depend on t

Defining a Conditional Density

- ▶ Work with random vector $\underline{x} = (X, Y) \sim F(x, y)$.
 - ▶ X and Y are random variables
 - ▶ x and y are realizations of the random variables
 - ▶ $F(x, y)$ is joint cumulative distribution
 - ▶ $f(x, y)$ is joint density function

Conditional Variables and Independence

- ▶ Conditional probability

- ▶ when $\Pr(\underline{x} \in B) > 0$,

$$\Pr(\underline{x} \in A | \underline{x} \in B) = \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

- ▶ Conditional distribution $F(y|x)$ (handles $\Pr(B) = 0$)

- ▶ Marginal distribution: $F_X(x) = \Pr(X \leq x)$

- ▶ $F(y|x)$ is $\Pr(Y \leq y)$ conditional on $X \leq x$

Defining a Conditional Density

- ▶ Independence: The random variables X and Y are independent if

$$F(x, y) = F_X(x) F_Y(y)$$

- ▶ If X and Y are independent, then

$$F(y|x) = F_Y(y)$$

and

$$F(x|y) = F_X(x)$$

- ▶ i.i.d means independent and identically distributed
- ▶ Conditional (mathematical, rational) expectation

$$E(Y|x) = \int_{-\infty}^{\infty} y dF(y|x) = \int_{-\infty}^{\infty} y f(y|x) dy.$$

Markov Property

- ▶ A particular conditional process is called a Markov chain.
- ▶ Markov Property: A stochastic process $\{x_t\}$ is said to have the Markov property if for all $k \geq 1$ and all t ,

$$Prob(x_{t+1}|x_t, x_{t-1}, \dots, x_{t-k}) = Prob(x_{t+1}|x_t) \quad (1)$$

- ▶ That is, the dependence between random events can be summarized exclusively with the previous event.
- ▶ This allows us to characterize this process with a Markov chain.
- ▶ Markov chains are a key way of characterizing stochastic events in our models.

Markov Chains

- ▶ For a stochastic process with the Markov property, we can characterize the process with a Markov chain.
- ▶ A time-invariant Markov chain is defined by the tuple:
 1. an n -dimensional state space of vectors $e_i, i = 1, \dots, n$,
 - ▶ where e_i is an $n \times 1$ vector where
 - ▶ the i th entry equals 1 and the vector contains 0s otherwise.
 2. a transition matrix P ($n \times n$), which records the conditional probability of transitioning between states
 3. a vector π_0 ($n \times 1$), that records the unconditional probability of being in state i at time 0.
- ▶ The key object here is P . Elements of this matrix are given by

$$P_{ij} = \text{Prob}(x_{t+1} = e_j | x_t = e_i) \quad (2)$$

- ▶ In other words, if you're in state i , this is the probability you enter state j .

Markov Chains

- ▶ Some assumptions on P and π_0 :

- ▶ For $i = 1, \dots, n$, P satisfies

$$\sum_{j=1}^n P_{ij} = 1 \quad (3)$$

- ▶ π_0 satisfies

$$\sum_{i=1}^n \pi_{0i} = 1 \quad (4)$$

- ▶ Where does this first property become useful?
- ▶ How would you calculate $\text{Prob}(x_{t+2} = e_j | x_t = e_i)$?

$$= \sum_{h=1}^n \text{Prob}(x_{t+2} = e_j | x_{t+1} = e_h) \text{Prob}(x_{t+1} = e_h | x_t = e_i) \quad (5)$$

$$= \sum_{h=1}^n P_{ih} P_{hj} = P_{ij}^{(2)} \quad (6)$$

Markov Chains

- ▶ This is also true in general:

$$\text{Prob}(x_{t+k} = e_j | x_t = e_i) = P_{ij}^{(k)} \quad (7)$$

- ▶ Why is this useful? We can use π_0 with this transition matrix to characterize the probability distribution over time:

$$\pi'_1 = \pi'_0 P \quad (8)$$

$$\pi'_2 = \pi'_0 P^2 \quad (9)$$

$$\vdots \quad (10)$$

- ▶ Thus, by knowing the initial distribution and the transition matrix, P , we know the distribution at time t

Stationary Distributions

- ▶ Where does this trend to over time?
- ▶ We know that the transition of the distribution takes the form $\pi'_{t+1} = \pi'_t P$.
- ▶ This distribution is stationary if

$$\pi_{t+1} = \pi_t \tag{11}$$

- ▶ (we will relax this to t large enough momentarily)
- ▶ This means that for a stationary distribution, π, P satisfy

$$\pi' = \pi' P \text{ or} \tag{12}$$

$$(I - P')\pi = 0 \tag{13}$$

- ▶ Anyone recognize this?

Stationary Distributions

$$\pi' = \pi' P \text{ or} \quad (14)$$

$$(I - P')\pi = 0 \quad (15)$$

- ▶ A lot of linearizing dynamic systems is about
 - ▶ finding eigenvectors with corresponding eigenvalues of less than 1 (non-explosive).
 - ▶ solving for initial conditions that are orthogonal to the explosive eigenvectors (i.e., the system does not explode).
- ▶ Intuitive refresher:
 - ▶ eigenvector: tells me the direction a system moves (i.e., distance traveled)
 - ▶ eigenvalue: tells me how many times it traveled since I last saw it.

Stationary Distributions

$$\pi' = \pi' P \text{ or} \tag{14}$$

$$(I - P')\pi = 0 \tag{15}$$

- ▶ It is useful to note (and will be useful when we think of linearized solution techniques), that
 - ▶ π is the (normalized) eigenvector of the stochastic matrix P .
 - ▶ In this case, the eigenvalue (root) is 1.

Asymptotically Stationary Distributions

- ▶ What about when $\pi_0 \neq \pi_t$? Can it still have a notion of stationarity?
- ▶ Yes. Asymptotic stationarity.
- ▶ Asymptotic stationarity:

$$\lim_{t \rightarrow \infty} \pi_t = \pi_\infty \quad (16)$$

- ▶ where $\pi'_\infty = \pi'_\infty P$
- ▶ Next, is this ending point unique?
- ▶ Does π_∞ depend on π_0 ?
- ▶ If not, π_∞ is an invariant or stationary distribution of P .
- ▶ This will be very useful when we talk about heterogeneous agents.

Some Examples

▶ Let's pick a simple initial condition: $\pi'_0 = [1 \ 0 \ 0]$.

▶ And a matrix

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} \quad (17)$$

▶ Now use Matlab to iterate.

Preliminaries

```
>> piMat = MMat'*piMat  
  
piMat =  
  
    0.9000  
    0.1000  
         0
```

Figure: First iteration

```
>> piMat = MMat'*piMat  
  
piMat =  
  
    0.8300  
    0.1500  
    0.0200
```

Figure: 2nd iteration

```
>> piMat = MMat^(100)*piMat  
  
piMat =  
  
    0.6154  
    0.2308  
    0.1538
```

Figure: First iteration

```
>> piMat = MMat^(1000)*piMat  
  
piMat =  
  
    0.6154  
    0.2308  
    0.1538
```

Figure: Grid of k values

- ▶ This distribution (P) is asymptotically stationary!
- ▶ Unique? Try $\pi'_0 = [0 \ 0 \ 1]$

Preliminaries

```
>> piMat = MMat'*piMat  
  
piMat =  
  
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Figure: First iteration

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Figure: First iteration

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piMat =  
  
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```

Figure: Grid of k values

- ▶ This distribution (P) is (probably) a unique invariant distribution.
- ▶ How would we prove this?

Ergodicity

- ▶ We would like to be able to replace conditional expectations with unconditional expectations.
- ▶ i.e., not indexed by time or initial conditions.
- ▶ Some preliminaries:
 - ▶ Invariant function: “a random variable $y_t = \bar{y}'x_t$ is said to be invariant if $y_t = y_0$, $t \geq 0$, for all realizations of x_t , $t \geq 0$ that occur with positive probability under (P, π) .”
- ▶ i.e., the state x can move around, but the outcome y_t stays constant at y_0 .

Ergodicity

- ▶ Ergodicity:
 - ▶ “Let (P, π) be a stationary Markov chain. The chain is said to be ergodic if the only invariant functions \bar{y} are constant with probability 1 under the stationary unconditional probability distribution π .”
- ▶ In other words, for any initial distribution, the only functions that satisfy the definition of an invariant function are the same.

Next Time

- ▶ More stochastic processes.
- ▶ Homework due in one week.