

# Macro II: Stochastic Processes II

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# Introduction

- ▶ Today: study a variety of stochastic processes that show up in macroeconomics.
- ▶ Then, discuss detrending data.

# Stochastic Processes

- ▶ Conditional expectations and linear projections
- ▶ White noise
- ▶ AR(1)
- ▶ MA(1)
- ▶ ARMA(p,q)
- ▶ Detrending data

# What is a Stochastic Process?

- ▶ Stochastic process is an infinite sequence of random variables  $\{X_t\}_{t=-\infty}^{\infty}$
- ▶  $j$ 'th autocovariance =  $\gamma_j = C(X_t, X_{t-j})$
- ▶ Strict stationarity: distribution of  $(X_t, X_{t+j_1}, X_{t+j_2}, \dots, X_{t+j_n},)$  does not depend on  $t$
- ▶ Covariance stationarity:  $\bar{X}_t$  and  $C(X_t, X_{t-j})$  do not depend on  $t$

## White noise

$$\{\varepsilon_t\}_{t=-\infty}^{\infty}$$

- ▶  $E(\varepsilon_t) = 0, \forall t$
- ▶  $V(\varepsilon_t) = \sigma_\varepsilon^2, \forall t$
- ▶  $C(\varepsilon_t, \varepsilon_{t-j}) = 0, \forall t, j \neq 0$

## First-order autoregressive (AR(1)) process

$$x_t = \alpha + \phi x_{t-1} + \varepsilon_t$$

- ▶  $\varepsilon_t$  is white noise and  $|\phi| < 1$  as required by stationarity
- ▶ By recursive substitution under stationarity

$$\begin{aligned}x_t &= \alpha + \varepsilon_t + \phi [\alpha + \phi x_{t-2} + \varepsilon_{t-1}] \\ &= \frac{\alpha}{1 - \phi} + \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}.\end{aligned}$$

- ▶  $E(x_t) = \alpha / (1 - \phi)$

# Moments

► Facts

$$V(aX + bY) = a^2V(X) + b^2V(Y) + 2abC(X, Y)$$

$$C(aX + bY, cX + dY) = acV(X) + bdV(Y) + (ad + bc)C(X, Y)$$

► Since the value of  $x_t$  can be expressed as

$$x_t = \frac{\alpha}{1 - \phi} + \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j},$$

► The variance of  $x_t$  is

$$V(x_t) = \sum_{j=0}^{\infty} (\phi^j)^2 \sigma_{\varepsilon}^2 = \frac{\sigma_{\varepsilon}^2}{1 - \phi^2}.$$

# Covariances

► Covariance

$$\begin{aligned}C(x_t, x_{t-1}) &= C(\alpha + \phi x_{t-1} + \varepsilon_t, x_{t-1}) \\ &= 0 + \phi V(X) + 0 = \phi \frac{\sigma_\varepsilon^2}{1 - \phi^2},\end{aligned}$$

$$\begin{aligned}C(x_t, x_{t-k}) &= C\left(\phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j \varepsilon_{t-j}, x_{t-k}\right) \\ &= \phi^k \frac{\sigma_\varepsilon^2}{1 - \phi^2} = \phi^k V(x_t).\end{aligned}$$

- Expectation: If  $\{\varepsilon_t\}$  is i.i.d. and  $\alpha = 0$ ,  $E(x_t | x_{t-k}) = \phi^k x_{t-k}$



# AR(p)

- ▶ Autoregressive function of  $p$  lagged  $x$ 's

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + \varepsilon_t$$

- ▶ Defining  $x_{t-j} = L^j x_t$ , we can rewrite an AR(p) process as

$$(1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p) x_t = \alpha + \varepsilon_t$$

- ▶ Stationarity condition: The roots of

$$1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p = 0$$

lie outside the unit circle ( $|z| > 1$  when real)

## AR(p)

- ▶ For AR(1),  $\phi < 1$  yields stationarity
- ▶ If stationarity holds, we can rewrite  $x_t$  as a function of infinitely lagged  $\varepsilon$ 's

## First-order moving average (MA(1)) process

$$x_t = \alpha + \varepsilon_t + \theta\varepsilon_{t-1}$$

- ▶  $\varepsilon_t$  is white noise
- ▶  $E(x_t) = \alpha$
- ▶  $V(x_t) = (1 + \theta^2) \sigma_\varepsilon^2$
- ▶  $C(x_t, x_{t-1}) = C(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-1} + \theta\varepsilon_{t-2}) = \theta\sigma_\varepsilon^2$
- ▶  $C(x_t, x_{t-k}) = C(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-k} + \theta\varepsilon_{t-k-1}) = 0, k > 1$

## MA cont'd

- ▶ Rewrite with lag operator as

$$x_t - \alpha = (1 + \theta L)\varepsilon_t$$

- ▶ When the root of

$$1 + \theta z = 0$$

lie outside unit circle (when  $|\theta| < 1$ )  $x_t$  is said to be invertible

$$\begin{aligned}\varepsilon_t &= \frac{(x_t - \alpha)}{(1 + \theta L)} \\ &= \frac{-\alpha}{1 + \theta} + \sum_{j=0}^{\infty} (-\theta)^j x_{t-j},\end{aligned}$$

- ▶ Express residual as infinite recursion of lagged  $x$ 's

## MA(q)

- ▶  $x_t$  is a function of  $q$  lagged residuals

$$x_t = \alpha + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \dots + \theta_q\varepsilon_{t-q}$$

- ▶ Rewriting with lag operator yields

$$(1 + \theta_1L + \theta_2L^2 + \dots + \theta_qL^q) \varepsilon_t = x_t - \alpha$$

- ▶ Invertibility condition is that the roots of

$$1 + \theta_1L + \theta_2L^2 + \dots + \theta_qL^q = 0$$

lie outside the unit circle ( $|z| > 1$  when real)

- ▶ If the invertibility condition holds, we can write the  $\varepsilon_t$  as an infinite function of lagged  $x$ 's

## ARMA process

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} \\ + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q}.$$

- ▶ Stationarity condition
  - ▶ Depends entirely on autoregressive coefficients
  - ▶ The roots of

$$1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p = 0,$$

must lie outside the unit circle ( $|\cdot| > 1$  when real)

# EX

- ▶ Example: For AR(1)

$$1 - \phi_1 z = 0$$

implying

$$z = \frac{1}{\phi_1}$$

need

$$|z| = |\phi_1^{-1}| > 1 \quad \text{requiring } |\phi_1| < 1$$

## Invertibility Condition

- ▶ The roots of

$$1 - \theta_1 z - \theta_2 z^2 - \dots - \theta_p z^p = 0,$$

must lie on or outside the unit circle ( $|\cdot| > 1$  when real)

- ▶ Example: For MA(1)

$$1 - \theta_1 z = 0$$

implying

$$z = \frac{1}{\theta_1}$$

need

$$|z| = |\theta_1^{-1}| \geq 1 \quad \text{requiring } |\theta_1| \leq 1,$$

where unity is included as a limit



# Stationarity and invertibility

- ▶ Stationarity and invertibility imply

- ▶ if  $\varepsilon_t$  is i.i.d., then  $\varepsilon_t$  is the innovation to  $x_t$

$$\varepsilon_t = x_t - E(x_t | x_{t-1}, x_{t-2}, \dots).$$

- ▶ knowledge of the entire sequences  $\{\varepsilon_{t-j}\}_{j=0}^{\infty}$  and  $\{x_{t-j}\}_{j=0}^{\infty}$  is equivalent

# Detrending

- ▶ Most of our (business cycle) models have nothing to say about trends in the data.
- ▶ i.e., these models generally don't explain growth.
- ▶ Need to detrend to get an appropriate data series.
- ▶  $Y_t = X_t + z_t$ 
  - ▶  $X_t$  is stationary
  - ▶  $z_t$  is a trend
- ▶ Trend stationary
  - ▶  $z_t$  is deterministic
  - ▶ example:  $z_t = \alpha t$

# Difference Stationary

- ▶ Difference stationary
  - ▶  $z_t$  is a random walk with  $\{\varepsilon_t\}$  a white noise process

$$z_t = z_0 + \sum_{j=1}^t \varepsilon_{t-j}$$

$$z_{t-1} = z_0 + \sum_{j=1}^{t-1} \varepsilon_{t-j}$$

$$z_t - z_{t-1} = \varepsilon_t$$

## Three approaches

- ▶ Linear detrending: Regress data on time and take residuals
- ▶ Use Hodrick-Prescott filter to separate data into a trend component and residuals and take residuals
- ▶ First difference the data

# HP Filter

- ▶ Most common approach: HP Filter.
- ▶ Idea: isolate low-frequency trends from high frequency cycles.
- ▶ Let  $\{y_t\}_{t=1}^{\infty}$  be a given series, where  $y_t = x_t + z_t$  as before.
- ▶  $x_t$  is the trend component,  $z_t$  is cyclical.
- ▶ Let  $\lambda$  be a parameter to be specified later, and consider the problem

$$\min_{x_1, x_2, \dots, x_T} \sum_{t=1}^T (y_t - x_t)^2 + \lambda \sum_{t=2}^{T-1} [(x_{t+1} - x_t) - (x_t - x_{t-1})]^2 \quad (1)$$

- ▶ What is going on here?
  - ▶ We are minimizing the cyclical component (first part), by moving the trend closer to the data.
  - ▶ But we are getting penalized ( $\lambda$ ) for making the trend too closely reflect the data.

## Next Time

- ▶ Discuss expectations in linear difference equations.
- ▶ Homework is due Thursday evening.