# Macro II: Difference Equations

Professor Griffy

UAlbany

Spring 2024

# Introduction

- Is everyone able to access the cluster?

- Today: review linear algebra/difference equations.

- Apply to time series/macroeconomics.

# A linear difference equation

- Simple first-order linear difference equation:

$$x_{t+1} = Ax_t + Cw_{t+1}$$

- We might think of $x_t$ as a vector of states (capital, assets, etc.)
- and $w_{t+1}$ as a vector of shocks.
- note that $w_{t+1}$ is not known at time-t.
- Thus, a stochastic difference equation.

# A linear difference equation

- Simple first-order linear difference equation:

$$x_{t+1} = Ax_t + Cw_{t+1}$$

- $w_{t+1}$ as a vector of shocks:
  - A1: iid $w_{t+1} \sim N(0, I)$
  - A2 (A1'):

$$E[w_{t+1}|J_t] = 0$$
$$E[w_{t+1}w'_{t+1}|J_t] = I$$
$$J_t = [w_t, ..., w_1, x_0]$$

  - A3 (A1"):

$$E[w_{t+1}] = 0$$
$$E[w_t w'_{t-j}] = I \text{ } if \text{ } j = 0 \text{ } and \text{ } 0 \text{ } otherwise$$

# A linear difference equation

- Simple first-order linear difference equation:

$$\begin{aligned} x_{t+1} &= Ax_t + Cw_{t+1} \\ y_t &= Gx_t \end{aligned}$$

- We can think of $y_t$ as some type of measurement equation.

- This is called a state-space formulation.

- We could also think of $y_t$ as a choice variable (more on this later).

# Eigenvalues and eigenvectors

- eigenvector: the direction a system moves.
- eigenvalue: the distance it moves in that direction.
- Simple first-order linear difference equation:

$$x_{t+1} = Ax_t + Cw_{t+1}$$
$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{C} \end{bmatrix} w_{t+1}$$

- This says that a subset $x_1$ of the state is always at its initial value, $x_{1,t} = x_{1,0}$.
- i.e., it has a unit eigenvalue: solution of $(A_{11} - 1)x_{1,0}$ is any $x_{1,0}$.
- For this to be *covariance stationary*, the eigenvalues of $\tilde{A}$ must all be less than 1.
- i.e., the solution to $(A - \lambda I)v = 0$ is $|\lambda| < 1$ or $v = 0$ and $\lambda = 1$.

# Lag operators: preliminaries

- Let **S** be a set of stochastic processes. Define the lag operator $L^n : \mathbf{S} \to \mathbf{S}$, $n$ an integer, by

$$L^n \{X_t\}_{t=-\infty}^{\infty} = \{X_{t-n}\}_{t=-\infty}^{\infty}.$$

- Lag operator is linear

$$L(aX_t + bL^n X_t) = (aL + bL^{n+1})X_t,$$

so that lag operations can be manipulated like polynomials.

# Preliminaries II

- Some geometry

- Because the lag operator is linear (everything nets out),

$$(1-\phi L^n)\left(\sum_{j=0}^{J}(\phi L^n)^j\right)X_t = \left(1-(\phi L^n)^{J+1}\right)X_t,$$

  and if $(\phi L^n)^{J+1} X_t$ and $\left(\sum_{j=0}^{J}(\phi L^n)^j\right) X_t$ "converge"—which might be true even if $|\phi| > 1$—we get

$$\frac{1}{1-\phi L^n}X_t = \left(\sum_{j=0}^{\infty}(\phi L^n)^j\right)X_t,$$

  the inverse of the operation $1-\phi L^n$

# Preliminaries III

- Suppose $X_t = c$, $\forall t$. Then

$$L^n c = L^n X_t = c.$$

- The lag operator does not shift information sets

$$L^n E_t \left( X_{t+j} \right) = E_t \left( X_{t+j-n} \right) \neq E_{t-n} \left( X_{t+j-n} \right).$$

# Linear difference equations again

- Another way to write it

$$\begin{aligned} & E_t\left(b_{t+1}\right)=\lambda b_t \\ \Leftrightarrow\ & E_t\left(\left(1-\lambda L\right) b_{t+1}\right)=0. \end{aligned}$$

- Rewrite this as

$$\begin{aligned} b_{t+1} & = \lambda b_t+\varepsilon_{t+1}, \\ \varepsilon_{t+1} & \equiv b_{t+1}-E_t\left(b_{t+1}\right). \end{aligned}$$

- As a forecast error, $\varepsilon_t$ forms a martingale difference sequence, i.e.

$$E_t\left(\varepsilon_{t+1}\right)=0$$

# LEDE II

- Generalize

$$
\begin{aligned}
b_{t+1} - c\lambda^{t+1} &= \lambda b_t - \lambda c \lambda^t + \varepsilon_{t+1}, \\
(1 - \lambda L)\left(b_{t+1} - c\lambda^{t+1}\right) &= \varepsilon_{t+1}, \\
b_{t+1} &= c\lambda^{t+1} + \frac{1}{1 - \lambda L}\varepsilon_{t+1},
\end{aligned}
$$

  where $c$ is a constant
- Solution tells us $b_t$ at any time, $t$.
- Goal: find (solve for) the set of admissible $\{\varepsilon_t\}$ and $c$
- Two approaches:
  - Backward (start at back) solution: follow sequence from past to now to find current value.
  - Foward (start forward) solution: start in future and work backwards to pin down path.

# Backward solution

- If time starts at $-\infty$, the backward solution (if well-defined) is

$$b_t = c\lambda^t + \sum_{j=0}^{\infty} \lambda^j \varepsilon_{t-j}.$$

- If time starts at 0, the backward solution is

$$b_t = b_0\lambda^t + \sum_{j=0}^{t-1} \lambda^j \varepsilon_{t-j},$$

  where $b_0$ is a (possibly) random variable

# Solution set restrictions

- Initial conditions:
  - $\{\varepsilon_t\}$ and $b_0$ are given.
  - i.e., Perfect foresight $\varepsilon_t = 0, \forall t$
- Non-explosiveness:

$$\lim_{j\to\infty} E_t\left(b_{t+j}\right) = 0, \quad \forall t,$$
$$\sup_t V\left(b_t\right) < \infty.$$

Note that

$$\begin{aligned} E_t\left(b_{t+2}\right) &= E_t\left(E_{t+1}\left(b_{t+2}\right)\right) \\ &= E_t\left(\lambda b_{t+1}\right) = \lambda\left(\lambda b_t\right), \\ \Rightarrow E_t\left(b_{t+j}\right) &= \lambda^j b_t. \end{aligned}$$

# Restrictions II

- If $|\lambda| < 1$, there are many $c$ and $\{\varepsilon_t\}$ where the non-explosiveness conditions do not restrict
- But if $|\lambda| \geq 1$, the only admissible solution is $\varepsilon_t = c = 0$, so that $b_t = 0, \forall t$
- Because if any deviation from steady-state, will explode over time.
- Note that if $|\lambda| \geq 1$, then $b_t$ cannot generally satisfy both an initial condition and a non-explosiveness condition

# Nonhomogeneous differential equations

- Wish to solve
$$E_t\left(x_{t+1}\right) = \lambda x_t + z_t,$$
where $\{z_t\}$ is a stochastic forcing process.

- Generalize by adding a bubble term
$$\begin{aligned} E_t\left(x_{t+1} - b_{t+1}\right) &= \lambda x_t + z_t - \lambda b_t \\ \Leftrightarrow E_t\left(\left(1 - \lambda L\right)\left(x_{t+1} - b_{t+1}\right)\right) &= z_t, \end{aligned}$$
where $b_{t+1}$ is a "bubble term" that solves
$$E_t\left(b_{t+1}\right) = \lambda b_t.$$

- i.e., a process unrelated to the fundamental term, $x_t$ (i.e. a bubble).

# General LEDE II

- The general problem is

$$\begin{aligned} x_{t+1} - b_{t+1} &= \lambda \left( x_t - b_t \right) + \widetilde{\eta}_{t+1} + z_t, \\ \widetilde{\eta}_{t+1} &\equiv \left( x_{t+1} - b_{t+1} \right) - E_t \left( x_{t+1} - b_{t+1} \right), \\ \left( 1 - \lambda L \right) \left( x_{t+1} - b_{t+1} \right) &= \widetilde{\eta}_{t+1} + z_t. \end{aligned} \tag{GP}$$

- Goal: find the set of admissible $\{\widetilde{\eta}_t\}$ and $\{b_t\}$
- $\tilde{\eta}_{t+1}$: expectational errors.
- $b_t$: bubble term (non-fundamental value).

# Backward solution

- $\{\widetilde{\eta}_t\}$ and $\{b_t\}$ cannot be identified separately
- If time starts at $-\infty$, backward solution (if well-defined) is

$$\begin{aligned} x_{t+1} &= \sum_{j=0}^{\infty} \lambda^j \left(z_{t-j} + \widetilde{\eta}_{t+1-j}\right) + b_{t+1} \\ &= \sum_{j=0}^{\infty} \lambda^j z_{t-j} + \widetilde{b}_{t+1}, \\ \widetilde{b}_{t+1} &\equiv b_{t+1} + \sum_{j=0}^{\infty} \lambda^j \widetilde{\eta}_{t+1-j}. \end{aligned}$$

- $\widetilde{b}_{t+1}$ is a bubble term
- Fundamental (sometimes called particular) solution is

$$x_{t+1} = \sum_{j=0}^{\infty} \lambda^j z_{t-j}$$

- i.e., must reflect sequence of shocks (stochastic forcing process).

# Backwards solution II

- If time starts at 0, the backward solution can be written as

$$x_{t+1} = \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \widetilde{\eta}_{t+1-j} + (x_0 - b_0)\,\lambda^{t+1} + b_{t+1},$$

which becomes

$$\begin{aligned} x_{t+1} &= \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \eta_{t+1-j} + x_0 \lambda^{t+1}, \\ \eta_t &\equiv \widetilde{\eta}_t + b_t - E_{t-1}(b_t) \\ &= x_t - E_{t-1}(x_t). \end{aligned}$$

- $x_t$ is stochastic, will have errors.
- $b_t$ is deterministic. Cannot be wrong or will be systematic.

# Forward solution

- First, rewrite

$$(1 - \lambda L)(x_{t+1} - b_{t+1}) = \widetilde{\eta}_{t+1} + z_t$$

as

$$\left(\frac{1 - \lambda L}{-\lambda L}\right)(-\lambda L)(x_{t+1} - b_{t+1}) = \widetilde{\eta}_{t+1} + z_t,$$

$$\left(1 - \lambda^{-1} L^{-1}\right)(x_t - b_t) = -\frac{1}{\lambda}(z_t + \widetilde{\eta}_{t+1}).$$

To ensure that $x_t$ is a function only of variables known at time $t$, write this as

$$E_t\left(\left(1 - \lambda^{-1} L^{-1}\right)(x_t - b_t)\right) = -\frac{1}{\lambda} E_t(z_t + \widetilde{\eta}_{t+1}).$$

# Forward solution II

- Invert the lag operator

$$E_t\left(x_t - b_t\right) = -\frac{1}{\lambda} E_t\left(\frac{1}{1-\lambda^{-1}L^{-1}}\left(z_t + \widetilde{\eta}_{t+1}\right)\right),$$

$$\begin{aligned} x_t &= -\frac{1}{\lambda} E_t\left(\sum\nolimits_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^j\left(z_{t+j} + \widetilde{\eta}_{t+j+1}\right)\right) + b_t, \\ &= -\frac{1}{\lambda} E_t\left(\sum\nolimits_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^j z_{t+j}\right) + b_t, \end{aligned}$$

because $E_t\left(\widetilde{\eta}_{t+j}\right) = 0, \forall j \geq 1$

- note: $\frac{1}{L}^j = L^{-j}$ subsumed into $z_{t+j}$ (bc negative exponent on lag operator equals lead operator)

# Forward solution III

- The fundamental (particular) solution is

$$x_t = -\frac{1}{\lambda} E_t \left( \sum_{j=0}^{\infty} \left( \frac{1}{\lambda} \right)^j z_{t+j} \right)$$

- Note that $\widetilde{\eta}_t$ depends only on the forcing process $z_t$

$$\widetilde{\eta}_t = -\frac{1}{\lambda} \left[ E_t \left( \sum_{j=0}^{\infty} \left( \frac{1}{\lambda} \right)^j z_{t+j} \right) - E_{t-1} \left( \sum_{j=0}^{\infty} \left( \frac{1}{\lambda} \right)^j z_{t+j} \right) \right], \forall t.$$

# Summing up

- Forward solution

$$x_t = -\frac{1}{\lambda} E_t \left( \sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j z_{t+j} \right) + b_t.$$

- Backward solution

$$x_{t+1} = \sum_{j=0}^{\infty} \lambda^j z_{t-j} + \tilde{b}_{t+1},$$

or

$$x_{t+1} = \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \eta_{t+1-j} + x_0 \lambda^{t+1}.$$

# Restrictions

- Initial conditions:
  - $x_0$ and $\{\widetilde{\eta}_t\}_{t=1}^{\infty}$ are directly given, for example with capital accumulation

$$\begin{aligned} & k_{t+1} = (1-\delta)\, k_t + i_t, \\ & k_0 \text{ given}, \\ & k_{t+1} - E_t\,(k_{t+1}) = 0,\ \forall t. \end{aligned}$$

- Non-Explosiveness (boundary condition):

$$\lim_{j\to\infty} E_t\,(x_{t+j}) = 0, \quad \forall t,$$

$$\sup_t V\,(x_t) < \infty.$$

# Solutions

- If $|\lambda| < 1$, for "well-behaved" $\{z_t\}$ (e.g, ARMA processes), one solves $(1 - \lambda L)^{-1}$ backwards to get

$$x_{t+1} = \sum_{j=0}^{\infty} \lambda^j z_{t-j} + \tilde{b}_{t+1},$$

with a large number of permissable $\left\{\tilde{b}_t\right\}$.

- But if $|\lambda| > 1$, for "typical" $\{z_t\}$ (e.g, ARMA processes), we must solve $(1 - \lambda L)^{-1}$ forward and set $b_t = 0$, so that

$$x_t = -\frac{1}{\lambda} E_t \left( \sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j z_{t+j} \right).$$

- If $|\lambda| > 1$, cannot satisfy both initial conditions and non-explosiveness

# Rule of Thumb

- If $|\lambda| < 1$, set

$$x_{t+1} = \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \eta_{t+1-j} + x_0 \lambda^{t+1}.$$

  and use initial conditions to pin down $x_0$ and $\{\eta_t\}$

- If $|\lambda| > 1$, set

$$x_t = -\frac{1}{\lambda} E_t \left( \sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j z_{t+j} \right).$$

- If $|\lambda| = 1$, consider case by case

# Next Time

- Discuss rational expectations and Lucas Critique.
- See my webpage for new homework.