Macro II

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Introduction

- Today: Market structure
- Complete markets:
  - Arrow-Debreu structure (time-0 contingent claims);
  - Arrow securities (sequentially traded one-period claims).
- Homework 3 due next Thursday.

Complete markets

- Individuals in the economy have access to a comprehensive set of risk-sharing contracts:
  - They can contract to insure against any event or sequence of events.
  - They write these contracts with other agents in the economy.
- Will lead to
  - Perfect risk sharing
  - i.e., representative agent.

Complete markets

- Define unconditional probability of sequence of shocks $s^t = [s_0, s_1, ..., s_t]$ to be $\pi_t(s^t)$.
- Assume there are $i = 1, ..., I$ consumers, each of whom receives a stochastic endowment $y_t^i(s^t)$.
- They purchase a consumption plan that stipulates consumption for any history of shocks and yields:

$$U_i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u_i[c_t^i(s^t)]\pi_t(s^t)$$

- These contracts yield expected lifetime utility, where $\lim_{s \to 0} u_i'(c) = +\infty$

Complete markets

- They purchase a consumption plan that stipulates consumption for any history of shocks and yields:

$$U_i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u_i[c_t^i(s^t)] \pi_t(s^t)$$

- And are subject to a feasibility constraint:

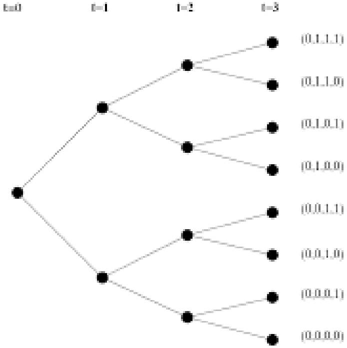
$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t) \; \forall \; t, \; s^t$$

- These contracts determine how to split resources at each t.

- i.e., they insure individuals ex-ante against income risk.

Contingent claims trading structure

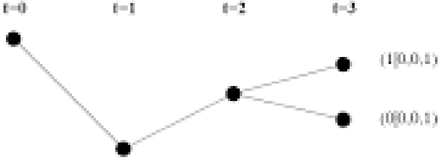
- Arrow-Debreu structure: contract at time $t = 0$ on *every* possible sequence of shocks.



- Each node represents a possible sequence of shocks.
- A consumption plan would specify consumption at each node at each time.

Sequential trading structure

- Arrow securities: re-contract at ever t given the history of shocks s^t.



- At $t = 2$, contract for two possible realizations.

Trading structure

- Arrow-Debreu structure: contract at time $t = 0$ on *every* possible sequence of shocks.
- Arrow securities: re-contract at ever t given the history of shocks s^t.
- Do these trading structure yield the same equilibrium allocation? Yes.
- Important property:
  - Under either structure, allocations are a function of the aggregate state only (& initial conditions).
  - i.e., allocation depends only on $\sum_{i=1}^{I} y_t^i(s^t)$
- Leads to representative agent structure.

Planner's Problem

- First, we will find the Pareto optimal allocation.
- i.e., the allocation from solving the Social Planner's problem:

$$\max_{c^i} W = \sum_{i=1}^{I} \lambda_i U_i(c^i)$$

- where λ_i is a "Pareto weight," i.e., how much Planner values individual i relative to others.
- Constrained maximization:

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \{ \sum_{i=1}^{I} \lambda_i \beta^t u_i(c_t^i) \pi_t(s^t) + \theta_t(s^t) \sum_{i=1}^{I} [y_t^i(s^t) - c_t^i(s^t)] \}$$

- i.e., maximize weighted expected utility subject to the feasibility constraint (multiplier θ)

Planner's Problem

- Constrained maximization:

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \{ \sum_{i=1}^{I} \lambda_i \beta^t u_i(c_t^i) \pi_t(s^t) + \theta_t(s^t) \sum_{i=1}^{I} [y_t^i(s^t) - c_t^i(s^t)] \}$$

- FOC in c_t^i:

$$\beta^t u_i'(c_t^i(s^t)) \pi_t(s^t) = \lambda_i^{-1} \theta_t(s^t)$$

- How is this allocated across consumers?

$$\frac{u_i'(c_t^i(s^t))}{u_1'(c_t^1(s^t))} = \frac{\lambda_1}{\lambda_i}$$
$$\rightarrow c_t^i(s^t) = u_i'^{-1}(\lambda_i^{-1} \lambda_1 u_1'(c_t^1(s^t)))$$

- Often, assume $\lambda_i = \lambda_1 \forall\ i \rightarrow c_t^i(s^t) = u_i'^{-1}(u_1'(c_t^1(s^t)))$

Planner's Problem

- Allocation:

$$c_t^i(s^t) = u_i'^{-1}(\lambda_i^{-1}\lambda_1 u_1'(c_t^1(s^t)))$$

- Sub into resource constraint:

$$\sum_i u_i'^{-1}(\lambda_i^{-1}\lambda_1 u_1'(c_t^1(s^t))) = \sum_i y_t^i(s^t)$$

- i.e., the resource allocation depends only on aggregate endowment and weights of each consumer.

Decentralized allocations

- We know that the optimal allocation is given by

$$\sum_i u_i'^{-1}(\lambda_i^{-1}\lambda_1 u_1'(c_t^1(s^t))) = \sum_i y_t^i(s^t)$$

- Can we achieve the same allocation under different trading regimes?

- Specifically, does the decentralized economy achieve the same allocation?

Consumer's problem

- Consumer's problem: maximize

$$U_i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u_i[c_t^i(s^t)] \pi_t(s^t)$$

- subject to

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

Consumer's problem

- Yields the following:

$$\beta^t u_i'[c_t^i(s^t)]\pi_t(s^t) = \mu_i q_t^0(s^t)$$
$$\frac{u_i'(c_t^i(s^t))}{u_1'(c_t^1(s^t))} = \frac{\mu_i}{\mu_1}$$

- which implies

$$\sum_i u_i'^{-1}(\mu_1^{-1}\mu_i u_1'(c_t^1(s^t))) = \sum_i y_t^i(s^t)$$

Competitive Equilibrium

Definition A competitive equilibrium is a price system $\{q_t^0(s^t)\}_{t=0}^{\infty}$ and allocation $\{c^{i*}\}_{i\in\mathcal{I}}$ such that

1. Given a price system, each individaul $i \in \mathcal{I}$ solves the following problem:

$$\{c_t^{i*}(s^t)\}_{t=0}^{\infty} = arg \max_{\{c_t^i(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\Big(c_t^i(s^t)\Big)\pi_t(s^t)$$

$$s.t. \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

2. On every history s^t at time t, market clears

$$\sum_{i\in\mathcal{I}} c_t^{i*}(s^t) = \sum_{i\in\mathcal{I}} y_t^i(s^t)$$

Rules out economies with externalities, incomplete markets, etc.

First Welfare Theorem

- First welfare theorem:
  Let c be a competitive equilibrium allocation. Then c is pareto efficient.

- Equivalence: Competitive equilibrium is a specific Pareto optimal allocation in which $\lambda_i = \mu_i^{-1}$.

Sequential trading

- Now, we will consider an economy with sequential trades.
- i.e., each period agents meet and trade state-contingent bonds
- Recall from asset pricing:

$$p_t = \beta E_t \left(\frac{u^{'}\left(d_{t+1}\right)}{u^{'}\left(d_t\right)} \left(p_{t+1} + d_{t+1}\right) \right)$$

- where the expectation is over realizations of s_{t+1}, which determines d_{t+1}.
- Price is determined by payout of asset across all different realizations.
- i.e., asset that provides good return across all realizations: expensive.

Market clearing

- Recall from asset pricing that the net bond position of the economy equaled zero.
- i.e., $\sum_i b^i_{t+1} = 0$.
- Same in this context.
- Some are borrowing and some are saving (in principle, if there were heterogeneity).
- This must net to zero.

Restriction: No Ponzi Schemes

- Must ensure that agents never take out too much debt.
- Natural debt limit:

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau)$$

- This is the amount that the agent could borrow and still commit to repay.
- Rules out Ponzi schemes.

Sequential problem

- Consumer's problem: maximize

$$U_i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u_i[c_t^i(s^t)] \pi_t(s^t)$$

- subject to

$$c_t^i + \sum_{s^{t+1}} Q_t(s_{t+1}|s^t) a_{t+1}^i(s_{t+1}, s^t) \leq y_t^i(s^t) + a_t^i(s^t)$$

$$-t + 1^i(s^{t+1}) \geq -A_{t+1}^i(s^{t+1})$$

- where Q_t is a pricing kernel: price of one unit of consumption given realization s_{t+1} and history s^t.

Sequential allocation

- Solving the previous problem yields the following Euler Equation:

$$Q_t(s_{t+1}|s^t) = \beta \left(\frac{u^{'}\left(c^i_{t+1}(s^{t+1})\right)}{u^{'}\left(c^i_t(s^t)\right)} \pi_t(s^{t+1}|s^t) \right)$$

- Same as the asset pricing specification from earlier.

- Taking the expectation of this expression across all possible realizations of s^{t+1} yields the price, Q.

Sequential Trading - Competitive Equilibrium

Definition A competitive equilibrium is a price system $\left\{\{Q_t(s_{t+1}|s^t)\}_{s_{t+1}\in S}\right\}_{t=0}^{\infty}$, an allocation $\left\{\left\{\tilde{c}_t^i(s^t),\ \{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}\in S}\right\}_{t=0}^{\infty}\right\}_{i\in\mathcal{I}}$, an initial distribution of wealth $\{a_0^i(s_0) = 0\}_{i\in\mathcal{I}}$, and a collection of natural borrowing limits $\left\{\left\{\{A_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}\in S}\right\}_{t=0}^{\infty}\right\}_{i\in\mathcal{I}}$ such that

1. Given a price system, an initial distribution of wealth, and a collection of natural borrowing limits, each individual $i \in \mathcal{I}$ solves the workers problem.
2. On every history s^t at time t, markets clear.

$$\sum_{i\in\mathcal{I}} c_t^i(s^t) = \sum_{i\in\mathcal{I}} y_t^i(s^t) \qquad \text{(Commodity market clearing)}$$

$$\sum_{i\in\mathcal{I}} a_{t+1}^i(s_{t+1}, s^t) = 0 \forall\ s_{t+1} \in S \qquad \text{(Asset market clearing)}$$

Equivalence of allocations

- Is this allocation also a time-0 trading allocation?
- Yes. Suppose that the pricing kernel takes the following form

$$q^0_{t+1}(s^{t+1}) = Q_t(s_{t+1}|s^t)q^0_t(s^t)$$
$$\frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = Q_t(s_{t+1}|s^t)$$

- That is, the price of 1 unit of consumption in period $t+1$ is the same regardless of whether you purchased that consumption last period or in period 0.
- When this holds, sequential allocation coincides with time-0 trading allocation, subject to initial distribution.
- Formal proof (check on your own): formal proof

Conclusion

- Midterm next Thursday (after break)!
- Check website for homework.

Equivalence of allocations

$$Q_t(s_{t+1}|s^t) = \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} \Rightarrow \beta \frac{u'\left(\tilde{c}^i_{t+1}(s^{t+1})\right)}{u'\left(\tilde{c}^i_t(s^t)\right)} \pi_t(s^{t+1}|s^t)$$

$$= \beta \frac{u'\left(c^{i*}_{t+1}(s^{t+1})\right)}{u'\left(c^{i*}_t(s^t)\right)} \pi_t(s^{t+1}|s^t)$$

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Guess for portfolio

On every history s^t at time t,

$$\tilde{a}^i_{t+1}(s_{t+1}, s^t) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|(s_{t+1},s^t)} \frac{q^0_\tau(s^\tau)}{q^0_{t+1}(s^{t+1})} \Big(c^{i*}_\tau(s^\tau) - y^i_\tau(s^\tau) \Big) \forall\ s_{t+1} \in S$$

Value of this portfolio expressed in terms of the date t, history s^t consumption good is $\sum_{s_{t+1}\in S} \tilde{a}^i_{t+1}(s_{t+1}, s^t) Q_t(s_{t+1}|s^t) =$

$$= \sum_{s_{t+1}\in S} \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|(s_{t+1},s^t)} \frac{q^0_\tau(s^\tau)}{q^0_{t+1}(s^{t+1})} \Big(c^{i*}_\tau(s^\tau) - y^i_t(s^\tau) \Big) Q_t(s_{t+1}|s^t)$$

$$= \sum_{s_{t+1}\in S} \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|(s_{t+1},s^t)} \frac{q^0_\tau(s^\tau)}{\cancel{q^0_{t+1}(s^{t+1})}} \Big(c^{i*}_\tau(s^\tau) - y^i_t(s^\tau) \Big) \frac{\cancel{q^0_{t+1}(s^{t+1})}}{q^0_t(s^t)}$$

$$= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} \frac{q^0_\tau(s^\tau)}{q^0_t(s^t)} \Big(c^{i*}_\tau(s^\tau) - y^i_t(s^\tau) \Big)$$

back

Verify portfolio

On history $s^0 = s_0$ at time $t = 0$, assume that $a_0^i(s_0) = 0$. Then

$$\tilde{c}_0^i(s_0) + \sum_{s_1 \in S} \tilde{a}_1^i(s_1, s_0) Q_1(s_1 | s_0) = y_0^i(s_0) + 0$$

$$\tilde{c}_0^i(s_0) + \sum_{\tau=1}^{\infty} \sum_{s^\tau | s_0} \frac{q_\tau^0(s^\tau)}{q_0^0(s_0)} \Big(c_\tau^{i*}(s^\tau) - y_t^i(s^\tau) \Big) = y_0^i(s_0) + 0$$

$$q_0^0(s_0) c_0^{i*}(s_0) + \sum_{\tau=1}^{\infty} \sum_{s^\tau | s_0} q_\tau^0(s^\tau) \Big(c_\tau^{i*}(s^\tau) - y_t^i(s^\tau) \Big) = q_0^0(s_0) y_0^i(s_0)$$

$$(\text{ if } \tilde{c}_0^i(s_0) = c_0^{i*}(s_0) \text{ })$$

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^{i*}(s^t)$$

Therefore, given $\tilde{c}_0^i(s_0) = c_0^{i*}(s_0)$, portfolio $\{\tilde{a}_1^i(s_1, s_0)\}_{s_1 \in S}$ is affordable.

back

Verify portfolio

On history s^t at time t, assume that
$\tilde{a}_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)} \left(c_\tau^{i*}(s^\tau) - y_\tau^i(s^\tau) \right)$. Then

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1} \in S} \tilde{a}_{t+1}^i(s_{t+1}, s^t) Q_t(s_{t+1}|s^t) = y_t^i(s^t)$$
$$+ \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)} \left(c_\tau^{i*}(s^\tau) - y_\tau^i(s^\tau) \right)$$

$$\tilde{c}_t^i(s^t) + \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)} \left(c_\tau^{i*}(s^\tau) - y_t^i(s^\tau) \right) = y_t^i(s^t)$$
$$+ \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)} \left(c_\tau^{i*}(s^\tau) - y_\tau^i(s^\tau) \right)$$

back

Verify portfolio

On history s^t at time t, assume that $\tilde{a}_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)} \left(c_\tau^{i*}(s^\tau) - y_\tau^i(s^\tau)\right)$. Then

$$q_t^0(s^t)c_t^{i*}(s^t) + \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} q_\tau^0(s^\tau)\left(c_\tau^{i*}(s^\tau) - y_t^i(s^\tau)\right) = q_t^0(s^t)y_t^i(s^t)$$

$$+ \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^0(s^\tau)\left(c_\tau^{i*}(s^\tau) - y_\tau^i(s^\tau)\right) \qquad (\text{ if } \tilde{c}_t^i(s^t) = c_t^{i*}(s^t)\ )$$

$$\rightarrow \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^0(s^\tau)\left(c_\tau^{i*}(s^\tau) - y_\tau^i(s^\tau)\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^0(s^\tau)\left(c_\tau^{i*}(s^\tau) - y_\tau^i(s^\tau\right.$$

Therefore, given $\tilde{c}_t^i(s^t) = c_t^{i*}(s^t)$, portfolio $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}\in S}$ is affordable.

back