Macro II

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Introduction

- So far: building tools to think about dynamic models.
- Now (and mostly rest of class):
  - Build on those tools to make more applicable to economics.
  - Use those tools to model the macroeconomy
- Today:
  - Introduce dynamic programming
- Homework due in one week.

Dynamic Programming

- Basic idea:
  - We can express macro models in a sequential form.
  - If we can write them *recursively*, we get access to more tools to solve them.
- We will start with a generic representation, give some important theorems, then discuss its use.

Sequential Problem

- We can broadly state most macro (and economics problems in general) as

$$sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, x_{t+1})$$

$$\text{s.t. } x_{t+1} \in \Gamma(x_t), \;\; t = 0, 1, 2, ...$$

$$x_0 \in X \text{ given}$$

- A solution tells us x_t at any time t.

Recursive Problem

- We want to write the sequential problem recursively

$$v(x) = sup_{y \in \Gamma(x)}[r(x,y) + \beta v(y)], \forall x \in X.$$

- *We can also find solutions to this problem that solve the sequential problem.*

- We can make statements about the existence and uniqueness of those solutions.

- These statements are often easier when expressed this way.

Some definitions

- Metric space: a set S together with a metric (distance function), $\rho : S \times S \Rightarrow R$, such that for all $x, y, z \in S$:
  1. $\rho(x, y) \geq 0$, equality iff $x = y$
  2. $\rho(x, y) = \rho(y, x)$
  3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$
- Complete metric space: A metric space (S, ρ) is complete if every Cauchy sequence converge to an element in S.
- Cauchy sequnce: a sequence $\{x_n\}|_{n=0}^{\infty}$ for which $\rho(x_n, x_m) < \epsilon$, any $\epsilon > 0$ for $n, m \geq N_\epsilon$
- i.e., a sequence that gets closer and closer together (think of a model converging to equilibrium).

Contraction Mapping

- If (S, ρ) is a complete metric space and $T : S \Rightarrow S$ is a contraction mapping with modulus β, then
  1. T has exactly one fixed point v in S, and
  2. for any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$, $n = 0, 1, 2, ...$

Blackwell's Sufficient Conditions

- Let $X \subseteq R^l$, and let $B(X)$ be a space of bounded functions $f : X \Rightarrow R$, with the sup norm. Let $T : B(X) \Rightarrow B(X)$ be an operator satisfying
  1. (monotonicity) $f, g \in B(X)$ and $f(x)g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$;
  2. (discounting) there exists some $\beta \in (0, 1)$ such that

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \text{ all } f \in B(X), a \geq 0, x \in X$$

Blackwell's Sufficient Applied

- Simple problem:

$$(Tv)(k) = \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v(y)\}$$

- Monotonicity: $f, g \in B(X)$ and $f(x)g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$;

- define $g(x) \geq v(x)$, then

$$\begin{aligned}(Tg)(k) &= \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta g(y)\} \\ &\geq \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v(y)\} \\ &= (Tv)(k)\end{aligned}$$

- To see, take difference. $g(y) \geq v(y) \rightarrow$ monotone.

Blackwell's Sufficient Applied

- Simple problem:

$$(Tv)(k) = \max_{0 \le y \le f(k)} \{U[f(k) - y] + \beta v(y)\}$$

- (discounting) there exists some $\beta \in (0,1)$ such that

$$[T(f + a)](x) \le (Tf)(x) + \beta a, \text{ all } f \in B(X), a \ge 0, x \in X$$

$$\begin{aligned}
(Tv)(k + a) &= \max_{0 \le y \le f(k)} \{U[f(k) - y] + \beta[v(y) + a]\} \\
&= \max_{0 \le y \le f(k)} \{U[f(k) - y] + \beta v(y) + \beta a\} \\
&= (Tv)(k) + \beta a
\end{aligned}$$

- Thus, contraction mapping. Existence and uniqueness.

Theorem of the Maximum

- Broadly stated, the problem we face is

$$(Tv)(x) = sup_y[F(x, y) + \beta v(y)]$$
$$\text{s.t. } y \text{ feasible given } x$$

- This is just a value function

- With a specified constraint.

Correspondences

- We will define a correspondence $\Gamma(x)$ as
  - a set of feasible values of $y \in Y$ for $x \in X$,
  - where X can be thought of as the set of possible states
  - and Y the set of possible choices.
- The easiest example: the budget constraint.
- There are many feasible choices,
- we will pick on the maximizes the return function.
- Argmax correspondence:
  - We define a policy function $G(x)$ as a correspondence, where
  - $G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$

Compact Sets

- A compact set is a set that
  1. is closed: contains all of its limit points.
  2. is bounded: all points are within a finite distance of each other.
- Useful: most often applied to choice sets.
- Means that choices are finite and feasible.

Upper and Lower Hemi-Continuity

- Two notions of continuity, (really) loosely:
  1. Upper hemi-continuity: any choice y is in the set $\Gamma(x)$ (closed).
  2. Lower hemi-continuity: nearby x are in $\Gamma(x)$.

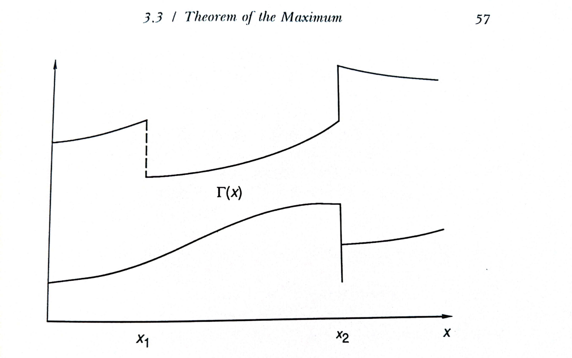

Figure 3.2

- Lower hemi-continuity: x_2 not lhc
- Upper hemi-continuity: x_1 not uhc

Upper and Lower Hemi-Continuity

- Upper hemi-continuity is useful:
  - Upper hemi-continuity preserves compactness:
  - if $C \subseteq X$ is compact and Γ is uhc,
  - $\Gamma(C)$ is compact.
- So if we place restrictions on X, our choice set is still in the correspondence.
- Allows our maximization problems to have solutions.
- If Γ is single-valued and uhc, it is continuous.

Theorem of the Maximum

- (conditions): Let $X \subseteq R^l$ and $Y \subseteq R^m$, let $f : X \times Y \Rightarrow R$ be a continuous function, and let $\Gamma : X \Rightarrow Y$ be a compact-valued and continuous correspondence.
- (implications): Then the function: $h : X \to R$ defined as $h(x) = \max_{y \in \Gamma(x)} f(x, y)$ and the correspondence $G : X \Rightarrow Y$ defined as $G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$ is
  1. nonempty,
  2. compact-valued, and
  3. upper hemi-continuous.
- Why is this useful?
  - under a few more assumptions (Γ is convex, f is strictly concave in y)
  - we can obtain the maxmized value of f using the control g.
  - and as a result, $h(x)$.

Stochastic Dynamic Programming

Returning to our initial definition, let r be the return function and u the control vector with a state that evolves by $x_{t+1} = h(x_t, u_t, \varepsilon_{t+1})$. The sequential problem looks like

$$\max_{\{u_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

$$\text{s.t.} \qquad x_{t+1} = h(x_t, u_t, \varepsilon_{t+1}) \ \forall t, \quad x_0 \text{ given.}$$

- where ε_t is some stochastic process ("shock") with a defined support and some distribution function $F(\varepsilon)$
- we usually take this to be independent and identically distributed *or* Markov.

The Equilibrium

What is the equilibrium in this environment? What are the equilibrium objects?

- A sequence $\{u_t\}_{t=0}^{\infty}$ for every possible sequence of realizations for ε's
- This is not so bad insofar as, at any given point in time, the problem has an infinite horizon and looks the same
- The above can be unwieldy, so we can instead find a *policy function* that tells the agent, at any point in time, what they should do given some observed x_t considering what they expect the ε's to be in the future

The Recursive Problem

Now let's translate this into a recursive problem.

$$V(x) = \max_u \Big\{ r(x,u) + \beta \mathbb{E}\Big[V\big(\underbrace{h(x,u,\varepsilon')}_{x'} \big) | x \Big] \Big\}$$

where $$\mathbb{E}\Big[V\big(h(x,u,\varepsilon')\big) | x \Big] \equiv \int_\xi V\big(h(x,u,\varepsilon')\big) dF(\varepsilon')$$

How do we solve this? The obvious way: FOCs:

$$\frac{dV(x)}{du} = 0 : \qquad r_2(x,u) + \beta \frac{d}{du} \mathbb{E}\Big[V\big(h(x,u,\varepsilon')\big) | x \Big] = 0$$

What allows us to pass the derivative through the expectation?

Differentiation under Integration

If the limits of integration *do not* depend on the control u, we can directly apply **Leibniz's rule** for differentiation under the integral (i.e., you just do it).

$$r_2(x,u) + \beta \mathbb{E}\left[\frac{dV(h(x,u,\varepsilon'))}{dx'} h_2(x,u,\varepsilon') \middle| x \right] = 0$$

Alas, another roadblock: we do not know what $dV(x')/dx'$ is. Now we'll want to apply the Envelope Theorem. That is, we'll want to find $dV(x)/dx$.

Envelope Theorem

- The envelope theorem always seems to be a source of confusion.
- It states (loosely) that when we are maximizing a value function V with a choice x, we can proceed as though all other choices are at their optimal values.
- Why is this important? Because in principle, u affects the choice of u'.

$$
\begin{aligned}
&r_2(x,u) + \beta\mathbb{E}\left[\frac{dV(h(x,u,\varepsilon'))}{dx'}h_2(x,u,\varepsilon')\Big|x\right] = 0 \\
&r_2(x,u) + \beta\mathbb{E}[(r_1(x',u') \\
&\qquad\qquad + (r_2(x',u') + \beta\mathbb{E}\frac{\partial V}{\partial u'}h_2(x',u',\epsilon''))\frac{\partial u'}{\partial x})h_2(x,u,\varepsilon')\big|x] = 0 \\
&r_2(x,u) + \beta\mathbb{E}[(r_1(x',u') \\
&\qquad\qquad + \cancel{(r_2(x',u') + \beta\mathbb{E}\frac{\partial V}{\partial u'}h_2(x',u',\epsilon''))}\frac{\partial u'}{\partial x})h_2(x,u,\varepsilon')\big|x] = 0
\end{aligned}
$$

- We can cancel future terms because we optimally pick u'
- i.e., we plug in $g(x)$ for u.

Envelope Theorem II

If the problem we are working with can be written in such a way such that the transition does not depend on x, this can be greatly simplified to

$$\frac{dV(x)}{dx} = r_1(x, u) \qquad \Longrightarrow \qquad \frac{dV(x')}{dx'} = r_1(x', u').$$

Plugging this back into the FOC gives the stochastic EE.

$$r_2(x, u) + \beta \mathbb{E}\left[r_1(x', u') h_2(x, u, \varepsilon') \middle| x \right] = 0$$

Now: return to neoclassical growth. Suppose that capital evolves according to $k' = (1 - \delta)k + a + \varepsilon$ (where ε is iid), and that there is full depreciation ($\delta = 1$).

Stochastic Neoclassical Growth

$$V(k,\varepsilon) = \max_{c,k'} \left\{ ln(c) + \beta \mathbb{E}\left[V(k',\varepsilon') \right] \right\} \qquad \text{s.t.} \qquad c = k^{\alpha} - k' + \varepsilon$$

$$\implies \quad V(k,\varepsilon) = \max_{k'} \left\{ ln(k^{\alpha} - k' + \varepsilon) + \beta \mathbb{E}\left[V(k',\varepsilon') \right] \right\}$$

The FOC is given by

$$\frac{1}{k^{\alpha} - k' + \varepsilon} = \beta \mathbb{E}\left[\frac{dV(k',\varepsilon')}{dk'} \right],$$

where we passed the derivative through the integral using Leibniz's rule.

Solving

Now for the Envelope Theorem.

$$\frac{dV(k,\varepsilon)}{dk} = \frac{\alpha k^{\alpha-1}}{k^\alpha - k' + \varepsilon} \qquad \Longrightarrow \qquad \frac{dV(k',\varepsilon')}{dk'} = \frac{\alpha k'^{\alpha-1}}{k'^\alpha - k'' + \varepsilon'}$$

Plugging this back into the FOC, we have the EE (which we can rewrite however we want).

$$\frac{1}{k^\alpha - k' + \varepsilon} = \beta \mathbb{E}\left[\frac{\alpha k'^{\alpha-1}}{k'^\alpha - k'' + \varepsilon'}\right]$$

$$\frac{1}{c} = \beta \mathbb{E}\left[\frac{\alpha k'^{\alpha-1}}{c'}\right]$$

Next Time

- Next: Permanent Income and Consumption Smoothing
- Homework due next Thursday.
- I may be out of town next Thursday. Will let you know whether virtual or canceled.