

Macro II

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Spring 2024

Introduction

- ▶ Today: reintroduce stochastic neoclassical growth model.
- ▶ Jumping off point for most modern macro models.
- ▶ Introduce solution techniques.

Model environment

- ▶ As an aside, since this is a straightforward model, let's talk about presentations.
- ▶ Make sure you tailor your talk to the audience.
- ▶ (Almost) Every macro presentation should have an environment slide.
- ▶ This details the following:
 1. preferences
 2. technology
 3. markets
- ▶ Next, what are the states and decisions of the agents?
- ▶ Then introduce the individual problem.
- ▶ We will go through each of these.

Consumer's Problem

- ▶ consumers in this economy maximize the expected value given by

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

- ▶ u is a bounded, continuous and strictly increasing utility function.
- ▶ $\beta \in (0, 1)$ is a discount factor.
- ▶ subject to

$$y_t \geq c_t + \underbrace{[k_{t+1} - (1 - \delta)k_t]}_{=i_t}$$

Resource Constraints

$$y_t \geq c_t + i_t$$
$$k_{t+1} = (1 - \delta)k_t + i_t$$

- ▶ An agent owns an amount $y_t \in \mathbb{R}_+ := [0, \infty)$ of consumption good at time t .
- ▶ Output can either be consumed or invested.
- ▶ When the good is invested, it is transformed one-for-one into capital.

Technology

$$y_{t+1} = f(\gamma_{t+1}, k_{t+1}), \gamma_{t+1} \sim \Phi$$

- ▶ $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is the production function which is increasing and continuous in k and γ .
- ▶ Production is stochastic, in that it depends on a shock γ_{t+1} realized at the end of the current period t .
- ▶ Calibration

$$\gamma_{t+1} = e^{\sigma \epsilon_{t+1}}, \epsilon_{t+1} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1), \sigma > 0$$

$$f(k_{t+1}, \gamma_{t+1}) = \gamma_{t+1} k_{t+1}^\alpha$$

$$\delta = 1$$

Optimization

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

$$s.t. \ y_t \geq c_t + k_{t+1} \ \forall t \quad (\text{Resource Constraint})$$

$$y_{t+1} = f(k_{t+1}, \gamma_{t+1}), \ \gamma_{t+1} \stackrel{i.i.d.}{\sim} \Phi \ \forall t \quad (\text{Technology})$$

$$c_t \geq 0 \ k_{t+1} \geq 0 \ \forall t \quad (\text{Non-negativity Constraint})$$

$$y_0 = \bar{y}_0 \ \text{given}$$

Sequential Problem (SP)

$$\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

$$s.t. y_{t+1} = f(k_{t+1}, \gamma_{t+1}), \gamma_{t+1} \stackrel{i.i.d.}{\sim} \Phi \forall t$$

$$y_t \geq c_t + k_{t+1} \forall t$$

$$y_0 = \bar{y}_0 \text{ given}$$

- ▶ Resource constraint holds with equality b/c $u' > 0$.
- ▶ y_t summarizes state of world at the start of each period.
- ▶ c_t is chosen by the agent each period after observing the state.

Functional Equation (FE)

(SP) is an infinite-dimensional optimization problem. Instead, find a time-invariant solution to functional equation:

$$v^*(y) = \max_{c \in [0, y]} \left\{ u(c) + \beta \int v^*(f(k', \gamma)) \phi(d\gamma) \right\}$$
$$y = c + k'$$
$$y' = f(k', \gamma)$$

Solution v^* , evaluated at $y = \bar{y}_0$, gives the value of the maximum in (SP).

Steady State

- ▶ Hard to characterize dynamics/solve model (find $c(t), k(t) \forall t$)
- ▶ Instead, characterize steady-state.
- ▶ $c = c' = c^*, k = k' = k^*$.
- ▶ pick $u(c) = \ln(c)$, $f(k, \gamma) = e^{\sigma\epsilon} k^\alpha$ and $\epsilon \sim N(0, 1), \sigma = 1$.
- ▶ then

$$\frac{1}{c} = \beta \mathbb{E}[(\alpha e^{\epsilon'} k'^{\alpha-1}) \frac{1}{c'}]$$

- ▶ In steady state:

$$\frac{1}{c^*} = \beta (\alpha \bar{\gamma} k^{*\alpha-1}) \frac{1}{c^*}$$

- ▶ We will solve for the *stochastic steady state*.
- ▶ i.e., the steady-state if the aggregate shock were at its mean.

Steady State

- ▶ This leaves us with capital:

$$1 = \beta(\alpha\bar{\gamma}k^{*\alpha-1})$$

$$k^* = \left(\frac{1}{\bar{\gamma}\alpha\beta}\right)^{\frac{1}{\alpha-1}}$$

$$k^* = (\bar{\gamma}\alpha\beta)^{\frac{1}{1-\alpha}}$$

- ▶ Now consumption from the budget constraint:

$$c^* + k^* = \bar{\gamma}k^{*\alpha}$$

$$c^* = \bar{\gamma}k^{*\alpha} - k^*$$

$$c^* = \bar{\gamma}(\bar{\gamma}\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\bar{\gamma}\alpha\beta)^{\frac{1}{1-\alpha}}$$

Dynamics

- ▶ Dynamics:

$$c' = \beta(\alpha\bar{\gamma}k'^{\alpha-1})c$$

$$k' = \bar{\gamma}k^\alpha - c$$

- ▶ We have two dynamic variables: c and k .
- ▶ The behavior of this system will depend on their dynamics.
- ▶ At steady-state:

$$1 = \frac{c'}{c} = \beta(\alpha\bar{\gamma}k^{\alpha-1})$$

$$1 = \frac{k'}{k} = \bar{\gamma}k^{\alpha-1} - \frac{c}{k}$$

- ▶ If both hold, we are in steady-state, if not, dynamics can vary.

Dynamics

- ▶ Dynamics:

$$c' = \beta(\alpha\bar{\gamma}k^{\alpha-1})c$$

$$k' = \bar{\gamma}k^{\alpha} - c$$

- ▶ Small c : second equation dictates that capital increases.
- ▶ Small k : first equation dictates that consumption increases.
- ▶ Reverse is true.

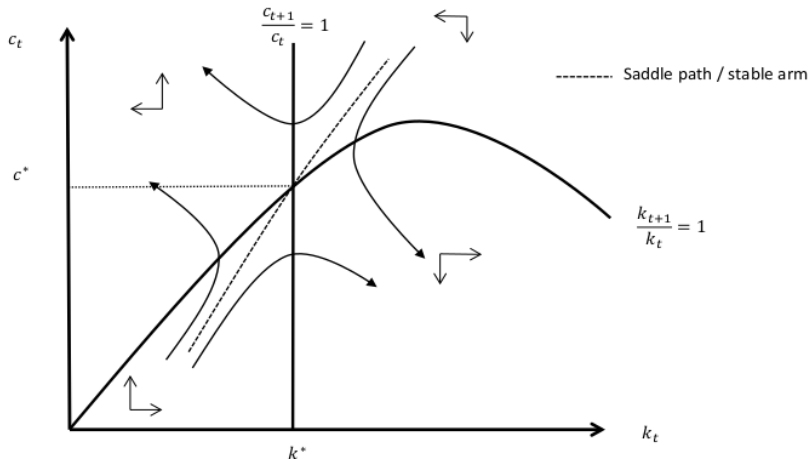
Phase Diagram

- Dynamics:

$$c' = \beta(\alpha\bar{\gamma}k'^{\alpha-1})c$$

$$k' = \bar{\gamma}k^\alpha - c$$

- From Eric Sim's notes:



Solving for dynamics

- ▶ Recall contraction mapping
- ▶ If T is a contraction mapping with modulus β , then
 1. there exists a unique fixed point v^* , and

$$v^* = Tv^*$$

$$v^*(y) = \max_{c \in [0, y]} \left\{ u(c) + \beta \int v^*(f(k', \gamma)) \phi(d\gamma) \right\}$$

$$y = c + k'$$

$$y' = f(k', \gamma)$$

2. for any v_0 and any $n \in \mathbb{N}$,

$$\rho(T^n v_0, v^*) \leq \beta^n \rho(v_0, v^*)$$

- ▶ Yields an is a policy function $\sigma^*(y) = \operatorname{argmax}\{v(y)\}$
- ▶ We can apply this in two ways: analytically or computationally.

Computation

How can we implement Bellman operator on our computer?

$$Tw(y) := \max_{c \in [0, y]} \left\{ u(c) + \beta \int \underbrace{w(f(k', \gamma))}_{\text{1. Approximation}} \phi(d\gamma) \right\}$$

2. Integration

3. Optimization

where w is a function that approximates v .

Approximation

- ▶ Approximate an analytically intractable real-valued function f with a computationally tractable function \hat{f}
- ▶ given limited information about f .
- ▶ Divide the **approximation domain** of the function into **finite number of sub-intervals** and approximate the original function in each of the intervals.
 - ▶ The points on the domain which separate the intervals are called grid points.
 - ▶ We use the value of the function at each grid point to approximate the original function.
- ▶ Another way to think about it: sampling from domain of the function at n nodes. As $n \rightarrow \infty$, $\hat{f} \rightarrow f$

Approximation II

In order to figure out Bellman operator, we need to approximate an analytically intractable real-valued function w .

$$Tw(y) := \max_{c \in [0, y]} \left\{ u(c) + \beta \int w(f(k', \gamma)) \phi(d\gamma) \right\}$$

► Interpolation

1. Determine an approximation domain of w .
2. Pick n (often evenly spaced) nodes, produces $n - 1$ intervals.
3. Approximate the original function w in each of the resulting intervals using a polynomial.

► Grid search

1. Determine an approximation domain of w .
2. Pick n nodes, produces $n - 1$ intervals.
3. Evaluate function at each node and pick maximum.

Integration

In order to figure out Bellman operator, we need to evaluate continuation value.

$$Tw(y) := \max_{c \in [0, y]} \left\{ u(c) + \beta \int w(f(k', \gamma)) \phi(d\gamma) \right\}$$

- ▶ One approach: Monte Carlo integration: Given a random sample of size n , $\{\gamma_i\}_{i=1}^n$

$$\frac{1}{n} \sum_{i=1}^n \frac{w(f(k', \gamma_i)) \cancel{\phi(\gamma_i)}}{\cancel{\phi(\gamma_i)}} \xrightarrow{p} \int w(f(k', \gamma)) \phi(\gamma) d\gamma$$

- ▶ Better approach: Gaussian Quadrature (for normally distributed shocks and from related families)

Optimization

- ▶ Find the minimum of some real-valued function of several real variables on a domain that has been specified.
 - ▶ Derivative methods: Newton's method, etc.
 - ▶ Derivative free: Golden section search, Grid search: pick maximizing node.
- ▶ Finding the global minimum can be challenging.
 - ▶ The function can have many local minima.
 - ▶ Curse of dimensionality & curvature of problem (when problems approach boundaries).

Conclusion

- ▶ Alternative: guess and verify (method of undetermined coefficients).
- ▶ We will cover this after returning to complete markets.
- ▶ Midterm in (about) two weeks.