

# Macro II

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# Announcements

- ▶ Today: Start discussing solution techniques.
- ▶ Focus on linearization & its problems.
- ▶ Midterm on Thursday! Will not cover today's lecture.
- ▶ HW4 due tonight.

# Motivation

- ▶ Models are hard to solve globally.
- ▶ Requires a lot of grid points, entails curse of dimensionality, takes a long time.
- ▶ A linearized system, by contrast, is easy to solve.
- ▶ Need to pick a place to linearize around.
- ▶ Pick the steady state.
- ▶ Underlying assumption: economy will stay close to the steady-state.

## Empirical Motivation

- ▶ Standard RBC: all fluctuations of hours worked on the *intensive* margin, i.e. average number of hours worked.
- ▶ Data: little fluctuation in average hours worked; lots of fluctuation in whether or not people are working (*extensive* margin).
- ▶ Standard RBC: missed badly on labor fluctuations (Frisch Elasticity, i.e. response of labor to change in wage too low).
- ▶ Solution: Modify model to have extensive margin with high Frisch Elasticity.
- ▶ Now: households pick the *probability* of working, but have to work a set number of hours.
- ▶ This is a *nonconvexity* in that it forces individuals to work either 0 or  $h$  hours.

## Hansen (1985)

- ▶ Neoclassical growth model with labor-leisure lottery.
- ▶ A social planner maximize the following:

$$E\left(\sum_{t=0}^{\infty} \beta^t [\ln(C_t) - \gamma H_t]\right) \quad (1)$$

- ▶ Subject to the following constraints:

$$Y_t = A_t K_t^\theta (\eta^t H_t)^{1-\theta} \quad (2)$$

$$\ln(A_t) = (1 - \rho)\ln(A) + \rho\ln(A_{t-1}) + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2) \quad (3)$$

- ▶ The goods market clears and capital evolves in a predetermined fashion.
- ▶ Here, we assume that per capita labor productivity grows at rate  $\eta$ .

## Equilibrium

- ▶ First step: detrend appropriate variables by per capita growth to get stationarity: i.e.  $y_t = Y_t/\eta^t$ .
- ▶ The system of equations that characterize the equilibrium are:

$$y_t = a_t k_t^\theta h_t^{1-\theta} \quad (4)$$

$$\ln(a_t) = (1 - \rho)\ln(A) + \rho\ln(a_{t-1}) + \epsilon_t \quad (5)$$

$$y_t = c_t + i_t \quad (6)$$

$$\eta k_{t+1} = (1 - \delta)k_t + i_t \quad (7)$$

- ▶ Combine FOC[c] and FOC[h]:

$$\gamma c_t h_t = (1 - \theta)y_t \quad (8)$$

- ▶ Euler Equation:

$$\frac{\eta}{c_t} = \beta E_t \left[ \frac{1}{c_{t+1}} \left( \theta \left( \frac{y_{t+1}}{k_{t+1}} \right) + 1 - \delta \right) \right] \quad (9)$$

## Solving for the Steady-State

$$\begin{aligned} \ln(a^*) &= (1 - \rho)\ln(A) + \rho\ln(a^*) \\ \Rightarrow \ln(a^*) &= \ln(A) \end{aligned} \tag{10}$$

Euler Equation:

$$\begin{aligned} \frac{\eta}{c^*} &= \beta E_t \left[ \frac{1}{c^*} \left( \theta \left( \frac{y^*}{k^*} \right) + 1 - \delta \right) \right] \\ \Rightarrow \frac{\eta}{\beta} &= \theta \frac{y^*}{k^*} + 1 - \delta \\ \Rightarrow k^* &= \left( \frac{\theta}{\frac{\eta}{\beta} - 1 + \delta} \right) y^* \end{aligned} \tag{11}$$

## Solving for the Steady-State

- ▶ Use the previous to solve for investment

$$\begin{aligned}\eta k^* &= (1 - \delta)k^* + i^* \\ \Rightarrow (\eta - 1 + \delta)k^* &= i^* \\ \Rightarrow i^* &= \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)y^*\end{aligned}\tag{12}$$

- ▶ FOC[c] and FOC[h]:

$$\begin{aligned}\gamma c^* h^* &= (1 - \theta)y^* \\ \Rightarrow \gamma \left[1 - \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)\right] y^* h^* &= (1 - \theta)y^* \\ \Rightarrow h^* &= \left(\frac{1 - \theta}{\gamma}\right) \left[1 - \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)\right]^{-1}\end{aligned}\tag{13}$$



## Solving for the Steady-State

- ▶ Finally, solve for output.

$$y^* = a^* k^{*\theta} h^{*1-\theta}$$

$$y^* = a^* \left( \left( \frac{\theta}{\frac{\eta}{\beta} - 1 + \delta} \right) y^* \right)^\theta \left[ \left( \frac{1-\theta}{\gamma} \right) \left[ 1 - \left( \frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta} \right) \right]^{-1} \right]^{1-\theta}$$

$$y^{*1-\theta} = a^* \left( \frac{\theta}{\frac{\eta}{\beta} - 1 + \delta} \right)^\theta \left[ \left( \frac{1-\theta}{\gamma} \right) \left[ 1 - \left( \frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta} \right) \right]^{-1} \right]^{1-\theta}$$

$$y^* = a^{*\frac{1}{1-\theta}} \left( \frac{\theta}{\frac{\eta}{\beta} - 1 + \delta} \right)^{\frac{\theta}{1-\theta}} \left[ \left( \frac{1-\theta}{\gamma} \right) \left[ 1 - \left( \frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta} \right) \right]^{-1} \right]^{1-\theta} \quad (14)$$

- ▶ All variables now a function of parameters.

## Steady-States

- ▶ In steady-state  $y_t = y_{t+1} = y^*$ .

$$\ln(a^*) = \ln(A) \quad (15)$$

$$k^* = \left(\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta}\right)y^* \quad (16)$$

$$i^* = \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)y^* \quad (17)$$

$$c^* = \left[1 - \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)\right]y^* \quad (18)$$

$$h^* = \left(\frac{1 - \theta}{\gamma}\right)\left[1 - \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)\right]^{-1} \quad (19)$$

$$y^* = a^{*\frac{1}{1-\theta}} \left(\frac{\theta}{\frac{\eta}{\beta} - 1 + \delta}\right)^{\frac{\theta}{1-\theta}} \left[\left(\frac{1 - \theta}{\gamma}\right)\left[1 - \left(\frac{\theta(\eta - 1 + \delta)}{\frac{\eta}{\beta} - 1 + \delta}\right)\right]^{-1}\right]^{1-\theta} \quad (20)$$

- ▶ These steady-states will be used for calibration/solving.

# Overview

- ▶ Broadly, two methods of solving models:
  1. Local linear methods.
  2. Global non-linear methods.
- ▶ Tradeoff: accuracy (global non-linear) for speed and simplicity (local linear).
- ▶ My preference: global methods (linear methods involve linearizing Euler Equation, distorting choices over risk).
- ▶ Here: Discuss log linearization and Blanchard and Kahn's Method.

# Local Linear Methods

- ▶ Log-linearize the system around the steady-state, then proceed.
- ▶ First have to solve the system for stability:
  1. Klein's Method (2000): Used for singular matrices.
  2. Sim's Method (2001): Used when it is unclear which variables are states and controls.
  3. Blanchard and Kahn's Method (1980): First solution method for rational expectations models.
- ▶ Here, we will use Blanchard and Kahn's Method.

## Log-Linearizing the System

We first wish to rewrite  $\tilde{x}_t = \ln(x_t) - \ln(x)$  in two convenient ways:

$$\tilde{x}_t = \ln\left(\frac{x_t}{x}\right)$$

Then, the first-order Taylor Approximation to this equation yields:

$$\tilde{x}_t \cong \tilde{x}_t(x) + \frac{\partial \tilde{x}_t}{\partial x_t}(x)(x_t - x)$$

$$\Rightarrow \tilde{x}_t \cong \ln(1) + \frac{1}{x}(x_t - x)$$

We can also rewrite the equation for  $\tilde{x}_t$  as

$$x_t = xe^{\tilde{x}_t} \tag{21}$$

## Log-Linearizing the System

From equilibrium conditions:

$$y_t = a_t k_t^\theta h_t^{1-\theta} \quad (22)$$

$$\Rightarrow \ln(y_t) = \ln(a_t) + \theta \ln(k_t) + (1 - \theta) \ln(h_t)$$

$$\ln(y) = \ln(a) + \theta \ln(k) + (1 - \theta) \ln(h)$$

$$\begin{aligned} \Rightarrow \tilde{y}_t = \ln(y_t) - \ln(y) &= \ln(a_t) + \theta \ln(k_t) + (1 - \theta) \ln(h_t) \\ &\quad - (\ln(a) + \theta \ln(k) + (1 - \theta) \ln(h)) \end{aligned}$$

$$\Rightarrow \tilde{y}_t = \tilde{a}_t + \theta \tilde{k}_t + (1 - \theta) \tilde{h}_t \quad (23)$$

## Log-Linearizing the System

$$\ln(a_t) = (1 - \rho)\ln(A) + \rho\ln(a_{t-1}) + \epsilon_t$$

$$\ln(a) = (1 - \rho)\ln(A) + \rho\ln(a)$$

$$\Rightarrow \tilde{a}_t = \rho\tilde{a}_{t-1} + \epsilon_t \quad (24)$$

## Log-Linearizing the System

$$y_t = c_t + i_t$$

$$\Rightarrow \tilde{x}_t \cong \ln(1) + \frac{1}{x}(x_t - x) = \left(\frac{x_t}{x} + 1\right)$$

$$\Rightarrow y(\tilde{y}_t + 1) = c(\tilde{c}_t + 1) + i(\tilde{i}_t + 1)$$

$$\tilde{y}_t = \frac{c}{y}\tilde{c}_t + \frac{i}{y}\tilde{i}_t$$



## Log-Linearizing the System

- Let  $\tilde{y}_t = \ln(y_t) - \ln(y^*)$ . Then, using Taylor Series approximations, the system characterizing the equilibrium becomes:

$$\tilde{y}_t = \tilde{a}_t + \theta \tilde{k}_t + (1 - \theta) \tilde{h}_t \quad (25)$$

$$\tilde{a}_t = \rho \tilde{a}_{t-1} + \epsilon_t \quad (26)$$

$$\left(\frac{\eta}{\beta} - 1 + \delta\right) \tilde{y}_t = \left[\frac{\eta}{\beta} - 1 + \delta - \theta(\eta - 1 + \delta)\right] \tilde{c}_t + \theta(\eta - 1 + \delta) \tilde{i}_t \quad (27)$$

$$\eta \tilde{k}_{t+1} = (1 - \delta) \tilde{k}_t + (\eta - 1 + \delta) \tilde{i}_t \quad (28)$$

$$\tilde{y}_t = \tilde{c}_t + \tilde{h}_t \quad (29)$$

$$0 = \frac{\eta}{\beta} \tilde{c}_t + E\left[\left(\frac{\eta}{\beta} - 1 + \delta\right)(\tilde{y}_{t+1} - \tilde{k}_{t+1}) - \frac{\eta}{\beta} \tilde{c}_{t+1}\right] \quad (30)$$

## Log-Linearizing the System

- ▶ We can now write the system as:

$$\Psi_1 \zeta_t = \Psi_2 \xi_t + \Psi_3 \tilde{a}_t \quad (\text{ME})$$

$$\Psi_4 E_t(\xi_{t+1}) = \Psi_5 \xi_t + \Psi_6 \zeta_t + \Psi_7 \tilde{a}_t \quad (\text{TE})$$

- ▶  $\zeta_t$  are static predetermined and nonpredetermined variables,  $[\tilde{y}_t, \tilde{i}_t, \tilde{h}_t]'$ .
- ▶  $\xi_t$  are dynamic predetermined and nonpredetermined variables,  $[\tilde{k}_t, \tilde{c}_t]'$ .
- ▶  $\tilde{a}_t$  is the technology process.
- ▶ Why is  $\tilde{c}_t$  among the dynamic variables?

# Matrices

$$\kappa = \eta/\beta - 1 + \delta$$

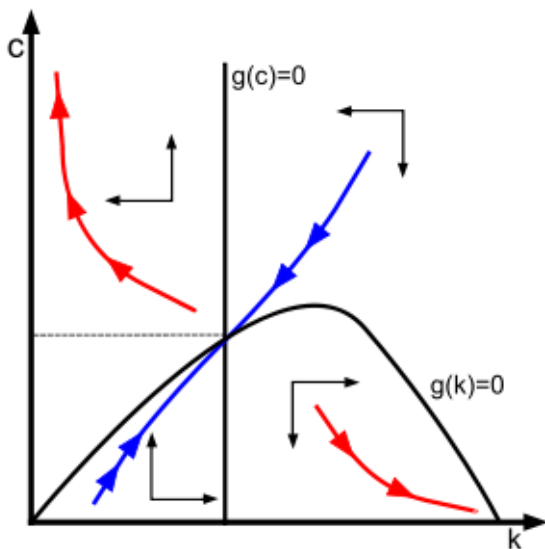
$$\lambda = \eta - 1 + \delta$$

$$\zeta_t = [\tilde{y}_t \quad \tilde{i}_t \quad \tilde{h}_t]' , \quad \xi_t = [\bar{k}_t \quad \bar{c}_t]'$$

$$\Psi_1 = \begin{bmatrix} 1 & 0 & \theta - 1 \\ \kappa & -\theta\lambda & 0 \\ 1 & 0 & 1 \end{bmatrix} , \quad \Psi_2 = \begin{bmatrix} \theta & 0 \\ 0 & \kappa - \theta\lambda \\ 0 & 1 \end{bmatrix} , \quad \Psi_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Psi_4 = \begin{bmatrix} \eta & 0 \\ \kappa & \eta/\beta \end{bmatrix} , \quad \Psi_5 = \begin{bmatrix} 0 & 0 & 0 \\ -\kappa & 0 & 0 \end{bmatrix} , \quad \Psi_6 = \begin{bmatrix} 1 - \delta & 0 \\ 0 & \eta/\beta \end{bmatrix} , \quad \Psi_7 = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## Solving the Model - Blanchard and Kahn (1980)



- ▶ Select  $\tilde{c}_0$  st the system isn't explosive (optimal control!).

## Solving the Model - Cont.

- ▶ Solve systems (TE and ME) so that  $\xi_{t+1}$  is only a function on  $\xi_t$  and  $\tilde{a}_t$ :

$$\Psi_1 \zeta_t = \Psi_2 \xi_t + \Psi_3 \tilde{a}_t \quad (31)$$

$$\Psi_4 E_t(\xi_{t+1}) = \Psi_5 \xi_t + \Psi_6 \zeta_t + \Psi_7 \tilde{a}_t \quad (32)$$

$$\Rightarrow \zeta_t = \Psi_1^{-1} [\Psi_2 \xi_t + \Psi_3 \tilde{a}_t]$$

- ▶ Plug into transition equation:

$$\begin{aligned} \Psi_4 E_t(\xi_{t+1}) &= \Psi_5 \xi_t + \Psi_6 \Psi_1^{-1} [\Psi_2 \xi_t + \Psi_3 \tilde{a}_t] + \Psi_7 \tilde{a}_t \\ \Rightarrow E_t(\xi_{t+1}) &= \Psi_4^{-1} [\Psi_5 + \Psi_6 \Psi_1^{-1} \Psi_2] \xi_t + \Psi_4^{-1} [\Psi_7 + \Psi_6 \Psi_1^{-1} \Psi_3] \tilde{a}_t \end{aligned} \quad (33)$$

- ▶ Desired result!

## Solving the Model - Cont.

- ▶ Having solved systems on previous slide so that  $\xi_{t+1}$  is only a function on  $\xi_t$  and  $\tilde{a}_t$ :

$$\begin{bmatrix} \tilde{k}_{t+1} \\ E_t(\tilde{c}_{t+1}) \end{bmatrix} = \Lambda^{-1} J \Lambda \begin{bmatrix} \tilde{k}_t \\ \tilde{c}_t \end{bmatrix} + E \tilde{a}_t \quad (34)$$

- ▶  $\Lambda^{-1} J \Lambda$  is the Jordan Decomposition.
- ▶ Subsume  $\Lambda$  into the model variables, denoted by hats:

$$\hat{c}_t = \Lambda_{12} \tilde{k}_t + \Lambda_{22} \tilde{c}_t \quad (35)$$

## Solving the Model - Cont.

- ▶ Subsume  $\Lambda$  into the model variables, denoted by hats.

$$\begin{bmatrix} \hat{k}_{t+1} \\ E_t(\hat{c}_{t+1}) \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + D\tilde{a}_t \quad (36)$$

$$E_t(\hat{c}_{t+1}) = J_2\hat{c}_t + D_2\tilde{a}_t \quad (37)$$

- ▶  $J_2 > 1 \rightarrow$  bad choice of  $c_t$  and this explodes.
- ▶ Solution: pick  $c_t$  so that it isn't a function of  $c_{t-1}$ !
- ▶ Rearranging:

$$\hat{c}_t = J_2^{-1}E_t(\hat{c}_{t+1}) - J_2^{-1}D_2\tilde{a}_t \quad (38)$$

## Solving the Model - Cont.

- ▶ Iterating on previous equation:

$$\hat{c}_{t+1} = J_2^{-1} E_t(\hat{c}_{t+2}) - J_2^{-1} D_2 \tilde{a}_{t+1} \quad (39)$$

$$\begin{aligned} \Rightarrow \hat{c}_t &= J_2^{-1} E_t(J_2^{-1} E_t(\hat{c}_{t+2}) - J_2^{-1} D_2 \tilde{a}_{t+1}) - J_2^{-1} D_2 \tilde{a}_t \\ \Rightarrow \hat{c}_t &= J_2^{-2} E_t(\hat{c}_{t+2}) - J_2^{-2} D_2 \rho \tilde{a}_t - J_2^{-1} D_2 \tilde{a}_t \end{aligned} \quad (40)$$

- ▶ Impose transversality condition (i.e.  $E_t(\hat{c}_{t+i}) = 0$  for large enough  $i$ ):

$$\Rightarrow \hat{c}_t = - \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 \rho \tilde{a}_t \quad (41)$$



## Solving the Model - Cont.

- ▶ Iterating on (33):

$$\hat{c}_t = \Lambda_{12}\tilde{k}_t + \Lambda_{22}\tilde{c}_t$$

$$\Rightarrow \Lambda_{22}\tilde{c}_t = -\Lambda_{12}\tilde{k}_t - \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 \rho \tilde{a}_t$$

- ▶ Solving this yields:

$$\Rightarrow c_t = -\Lambda_{22}^{-1}\Lambda_{12}\tilde{k}_t + (1/\Lambda_{22})\left(\frac{D_2}{\rho - J_2}\right)\tilde{a}_t \quad (42)$$

- ▶ The system will now be saddle-path stable.

## Next Time

- ▶ Midterm!
- ▶ Start value function iteration next week.
- ▶ New homework next week.