AECO 701 Midterm

Name:
UAlbany ID:

1 [40] **Neoclassical Growth with Vintage Capital.** Consider an economy with mass one of identical consumers whose preferences are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t ln(c_t)$$

where β discounts future utility. Consumers supply one unit of labor inelastically. In this economy, there are two sectors, one that employs "vintage" capital and one that employs "modern" capital. Aggregate productivity grows at rate γ, but is subject to idiosyncratic shocks:

$$A_{t+1} = (1+\gamma)A_t + \epsilon_t, \quad \epsilon \sim N(0, \sigma_\epsilon^2)$$

where $\gamma > 0$. Following the period, "vintage" capital depreciates completely. In other words, capital only lasts for two periods, but does not depreciate at all during its lifespan. Additionally, capital is paired to the aggregate productivity during its first period of use. Production in the "vintage" sector is given by the following:

$$Y_t^v = A_{t-1} k_{t-1}^\alpha (L_t^v)^{1-\alpha}$$

Modern Sector:

$$Y_t^m = A_t k_t^\alpha (L_t^m)^{1-\alpha}$$

The superscript v denotes output from the vintage sector, while the superscript m denotes production from the modern sector. Labor can be freely allocated between either of the two sectors. Aggregate resources are equal to the sum of output from both sectors.

a [20] Write the planner's problem in recursive form, being careful to note the state variables and choice variables.

Answer:

$$V(A_v, A_m, k_v, k_m) = \max_{c, L_v, L_m, k'_m} ln(c) + \beta E(V(A'_v, A'_m, k'_v, k'_m))$$

$$\begin{aligned}
\text{s.t. } c + k'_m = & \quad Y_v + Y_m \qquad (1) \\
Y_v = & \quad A_v k_v^\alpha L_v^{1-\alpha} \\
Y_m = & \quad A_m k_m^\alpha L_m^{1-\alpha} \\
L_m + L_v = & \quad 1 \\
K'_v = & \quad K_m \\
A'_v = & \quad A_m \\
A'_m = & \quad (1+\gamma)A_m + \epsilon, \quad \epsilon \sim N(0, \sigma_\epsilon^2)
\end{aligned}$$

b [15]Write the maximization problem and solve for the Euler Equation* and relationship between labor in both markets.

Answer:

$$V(A_v, A_m, k_v, k_m) = \max_{c, L_v, L_m, k'_m} ln(c) + \beta E(V(A'_v, A'_m, k'_v, k'_m)) - \lambda[c + k'_m - A_v k_v^\alpha L_v^{1-\alpha} - A_m k_m^\alpha (1-L_v)^{1-\alpha}]$$

$$\frac{\partial V}{\partial c} = \frac{1}{c} - \lambda \Rightarrow \frac{1}{c} = \lambda$$

$$\frac{\partial V}{\partial L_v} = -\lambda[-(1-\alpha)A_v k_v^\alpha L_v^{-\alpha} + (1-\alpha)A_m k_m^\alpha (1-L_v)^{-\alpha}]$$

$$\Rightarrow \frac{L_v}{1-L_v}^{-\alpha} = \frac{A_m}{A_v}\frac{k_m}{k_v}^{\alpha}$$

$$\Rightarrow \frac{L_m}{L_v} = (\frac{A_m}{A_v})^{\frac{1}{\alpha}}\frac{k_m}{k_v}$$

$$\frac{\partial V}{\partial k'_m} = -\lambda + \beta \frac{\partial V'}{\partial k'_v}$$

$$\Rightarrow \frac{\partial V}{\partial k'_m} = -\lambda + \beta\lambda'\alpha A'_v(\frac{L'_v}{k'_v})^{1-\alpha} \Rightarrow -\lambda = \beta E[\lambda'\alpha A'_v(\frac{L'_v}{k'_v})^{1-\alpha}]$$

$$\Rightarrow \frac{1}{c} = E[\frac{1}{c'}\alpha A'_v(\frac{L'_v}{k'_v})^{1-\alpha}]$$

c [5] Show the conditions under which the planner would assign more labor to the vintage sector than the modern sector. Take the ratio of capital to be exogenous.

From equation* (9), we know how the ratio of labor is decided. Substituting in for aggregate productivity in the ratio yields:

$$\frac{L_m}{L_v} = (\frac{(1+\gamma)A_v + \epsilon}{A_v})^{\frac{1}{\alpha}}\frac{k_m}{k_v}$$

$$\Rightarrow 1 \geq (\frac{(1+\gamma)A_v + \epsilon}{A_v})^{\frac{1}{\alpha}}\frac{k_m}{k_v}$$

$$\Rightarrow \frac{k_v}{k_m} \geq (\frac{(1+\gamma)A_v + \epsilon}{A_v})^{\frac{1}{\alpha}}$$

2 [25] **Contraction Mapping Theorem.** Blackwell's Sufficiency Conditions for a contraction are given by the following:

T is monotone if for $f(x) \leq g(x)$ $\forall x \in X$, then

$$Tf(x) \leq Tg(x) \quad \forall x \in X$$

T discounts if for some $\beta \in (0,1)$ and any $a \in \mathbb{R}_+$

$$T(f+a)(x) \leq Tf(x) + \beta a \quad \forall x \in X$$

a [25] Prove that the Bellman Operator satisfies both of these conditions.

Monotonicity:
Let $f(x) \leq g(x)$. Then

$$Tf(x) = h(x,y) + \beta f(x)$$
$$Tg(x) = h(x,y) + \beta g(x)$$

Taking the difference of these two yields:

$$\begin{aligned} Tf(x) - Tg(x) &= h(x,y) + \beta f(x) - [h(x,y) + \beta g(x)] \\ &= \beta(f(x) - g(x)) \end{aligned}$$

Since $f(x) \leq g(x)$, we know that $Tf(x) \leq Tg(x)$. Thus, the Bellman Operator is monotonic.
Discounting:
Let f and h be functions. Then

$$T(f+a)(x) \leq Tf(x) + \beta a$$
$$T(f+a)(x) = h(x,y) + \beta(f(x) + a)$$
$$\Rightarrow Tf(x) + \beta a = h(x,y) + \beta f(x) + \beta a$$

These are equivalent, so we see that β discounts.

3 [35] **Guess and Verify.** Consider an economy with an infinitely-lived representative consumer. Lifetime utility is given by

$$\sum_{t=0}^{\infty} \beta^t(\gamma ln(c_t) + (1-\gamma)ln(1-l_t))$$

Where $\beta, \gamma \in (0,1)$. Each individual is endowed with 1 unit of time each period, which can be split between working and leisure. The resource constraint is as follows:

$$c_t + k_{t+1} \leq \theta k_t^{\alpha} l_t^{1-\alpha}$$

Where $\theta > 0$ and $\alpha \in (0,1)$.

a [15] Write down the planner's problem.

Answer:

The planner's problem is given by the following:

$$V(k) = \max_{c,k',l}(\gamma ln(c) + (1-\gamma)ln(1-l)) + \beta V(k')$$

$$\text{s.t. } c + k \leq \theta k^{\alpha} l^{1-\alpha}$$

b [20] Guess that the value function takes the form $a_0 + a_1 ln(k)$. Assume that labor and leisure are constants and solve for a_0 and a_1, as well as policy functions for consumption, leisure, labor, and capital tomorrow (c(k), l(k), k'(k)).

Answer:

If we guess that the value function takes the form $V(k) = a_0 + a_1 ln(k)$, we can express the problem in the following way:

$$V(k) = \max_{c,k'}(\gamma ln(c) + (1-\gamma)ln(1-\bar{l})) + \beta a_0 + \beta a_1 ln(k') - \lambda[c + k' - \theta k^{\alpha}\bar{l}^{1-\alpha}]$$

$$\frac{\partial V}{\partial c} = \frac{1}{c} - \lambda = 0 \Rightarrow \frac{1}{c} = \lambda$$

$$\frac{\partial V}{\partial k'} = \frac{\beta a_1}{k'} - \lambda \Rightarrow \frac{\beta a_1}{k'} = \lambda$$

Plugging in for the budget constraint yields:

$$\frac{\beta a_1}{\theta k^{\alpha}\bar{l}^{1-\alpha} - c} = \frac{1}{c}$$

$$\Rightarrow c = \frac{1}{1 + a_1\beta}\theta k^{\alpha}\bar{l}^{1-\alpha}$$

$$\Rightarrow k' = \frac{a_1\beta}{1 + a_1\beta}\theta k^{\alpha}\bar{l}^{1-\alpha}$$

Combining this yields the following Bellman Equation*:

$$V(k) = (\gamma ln(\frac{1}{1 + a_1\beta}\theta k^{\alpha}\bar{l}^{1-\alpha}) + (1-\gamma)ln(1-\bar{l})) + \beta a_0 + \beta a_1 ln(\frac{a_1\beta}{1 + a_1\beta}\theta k^{\alpha}\bar{l}^{1-\alpha})$$

Plugging in our guess for the value function gives us:

$$a_0 + a_1 ln(k) = (\gamma ln(\frac{1}{1+a_1\beta}\theta k^\alpha \bar{l}^{1-\alpha}) + (1-\gamma)ln(1-\bar{l})) + \beta a_0 + \beta a_1 ln(\frac{a_1\beta}{1+a_1\beta}\theta k^\alpha \bar{l}^{1-\alpha})$$

We can separate this into a constant term and a term involving ln(k):

$$a_0 + a_1 ln(k) = (\gamma\alpha + a_1\alpha\beta)ln(k) + (1-\gamma)ln(1-\bar{l})) + \beta a_0 + \beta a_1 ln(\frac{a_1\beta}{1+a_1\beta}\theta\bar{l}^{1-\alpha}) + \gamma ln(\frac{1}{1+a_1\beta}\theta\bar{l}^{1-\alpha})$$

We can now solve for a_1 (this is called the method of undetermined coefficients, or guess and verify):

$$a_1 ln(k) = (\gamma\alpha + a_1\alpha\beta)ln(k)$$

$$\Rightarrow a_1 - a_1\alpha\beta = \gamma\alpha$$

$$\Rightarrow a_1 = \frac{\gamma\alpha}{1-\alpha\beta}$$

We can now solve for labor and leisure:

$$\frac{\partial V}{\partial l} = -\frac{1-\gamma}{1-l} - \lambda(1-\alpha)(\frac{k}{l})^\alpha \Rightarrow \frac{1-\gamma}{1-l} = \lambda(1-\alpha)(\frac{k}{l})^\alpha$$

$$\Rightarrow \frac{1-\gamma}{1-l} = \frac{1}{\frac{1}{1+a_1\beta}\theta k^\alpha l^{1-\alpha}}(1-\alpha)(\frac{k}{l})^\alpha$$

$$\Rightarrow \frac{1-\gamma}{1-l} = (1+a_1\beta)\frac{1}{\theta l}(1-\alpha)$$

$$\Rightarrow (1-\gamma)\theta l = (1-l)(1+a_1\beta)(1-\alpha)$$

$$\Rightarrow ((1-\gamma)\theta + (1+a_1\beta)(1-\alpha))l = (1+a_1\beta)(1-\alpha)$$

$$\Rightarrow l^* = \frac{(1+a_1\beta)(1-\alpha)}{((1-\gamma)\theta + (1+a_1\beta)(1-\alpha))}$$

Thus, we have the policy function for leisure strictly in terms of parameters. Now, we can turn to consumption as a function of the state:

$$c = \frac{1}{1+a_1\beta}\theta k^\alpha l^{*1-\alpha}$$

And next period's capital as a function of the state:

$$\Rightarrow k' = \frac{a_1\beta}{1+a_1\beta}\theta k^\alpha l^{*1-\alpha}$$

Finally, we can also solve for a_0, from equation* (33):

$$a_0 = (1-\gamma)ln(1-l^*)) + \beta a_0 + \beta a_1 ln(\frac{a_1\beta}{1+a_1\beta}\theta l^{*1-\alpha}) + \gamma ln(\frac{1}{1+a_1\beta}\theta l^{*1-\alpha})$$

$$\Rightarrow (1-\beta)a_0 = (1-\gamma)ln(1-l^*)) + \beta a_1 ln(\frac{a_1\beta}{1+a_1\beta}\theta l^{*1-\alpha}) + \gamma ln(\frac{1}{1+a_1\beta}\theta l^{*1-\alpha})$$

$$\Rightarrow a_0 = \frac{(1-\gamma)ln(1-l^*)) + \beta a_1 ln(\frac{a_1\beta}{1+a_1\beta}\theta l^{*1-\alpha}) + \gamma ln(\frac{1}{1+a_1\beta}\theta l^{*1-\alpha})}{1-\beta}$$