AECO 701 Midterm Answers

A

Name:
UAlbany ID:

Cake Eating Problem. Tom Hanks is the leader of a small island. He and his volleyball have a limited number of resources and face the following infinite horizon problem:

$$\sum_{t=0}^{\infty} \beta^t ln(c_t) \tag{1}$$

Where $\beta \in (0, 1)$. The resource constraint is as follows:

$$c_t + k_{t+1} \leq k_t \tag{2}$$

where $c_t, k_{t+1} \geq 0$, and $k_0 > 0$ is given. Here, k_t is the amount of cake remaining.

a [20] Write the problem recursively and solve for the Euler Equation. Answer:
We can write the problem recursively as

$$V(k) = \max_{c,k'} ln(c) + \beta V(k') \tag{3}$$

$$\text{s.t. } c + k' = k \tag{4}$$

$$\tag{5}$$

The Euler Equation is standard:

$$\frac{1}{c} = \beta \frac{1}{c'} \tag{6}$$

b [30] Write out Blackwell's Sufficient Conditions and show that the recursive formulation of this problem satisfies Blackwell's Sufficient Conditions. Answer:
Blackwell's Sufficient Conditions for a contraction are given by the following:
T is monotone if for $f(x) \leq g(x)$ $\forall x \in X$, then

$$Tf(x) \leq Tg(x) \quad \forall x \in X \tag{7}$$

T discounts if for some $\beta \in (0, 1)$ and any $a \in \mathcal{R}_+$

$$T(f + a)(x) \leq Tf(x) + \beta a \quad \forall x \in X \tag{8}$$

This is just a straightfoward application of Blackwell's Sufficient Conditions. A general proof is given below
Monotonicity:
Let $f(x) \leq g(x)$. Then

$$Tf(x) = h(x, y) + \beta f(x)$$

$$Tg(x) = h(x, y) + \beta g(x)$$

Taking the difference of these two yields:

$$\begin{aligned} Tf(x) - Tg(x) &= h(x,y) + \beta f(x) - [h(x,y) + \beta g(x)] \\ &= \beta(f(x) - g(x)) \end{aligned}$$

Since $f(x) \leq g(x)$, we know that $Tf(x) \leq Tg(x)$. Thus, the Bellman Operator is monotonic.
Discounting:
Let f and h be functions. Then

$$T(f+a)(x) \leq Tf(x) + \beta a$$

$$T(f+a)(x) = h(x,y) + \beta(f(x) + a)$$

$$\Rightarrow Tf(x) + \beta a = h(x,y) + \beta f(x) + \beta a$$

These are equivalent, so we see that β discounts.

c [30] Guess that the value function takes the form $V(k) = a_0 + a_1 ln(k)$. Solve for a_0 and a_1, find the optimal policy functions for k' and c, and then verify your guess.

Answer:
Guessing that the value function takes the form $V(k) = a_0 + a_1 ln(k)$ and plugging this into the recursive problem yields

$$RHS = \max_{k'} ln(k - k') + \beta[a_0 + a_1 ln(k')]$$

Taking the derivative of this with respect to k' yields

$$\begin{aligned} FOC[k'] &= -\frac{1}{k-k'} + \beta a_1 \frac{1}{k'} = 0 \\ \frac{1}{k-k'} &= \beta a_1 \frac{1}{k'} \\ k' &= \beta a_1 (k - k') \\ k' + \beta a_1 k' &= \beta a_1 k \\ (1 + \beta a_1) k' &= \beta a_1 k \\ k' &= \frac{\beta a_1 k}{1 + \beta a_1} \end{aligned}$$

The decision rule for c is straightforward:

$$\begin{aligned} c &= k - \frac{\beta a_1 k}{1 + \beta a_1} \\ &= (1 - \frac{\beta a_1}{1 + \beta a_1})k \\ &= \frac{1}{1 + \beta a_1} k \end{aligned}$$

Now, we can plug in these decision rules to solve for a_0 and a_1:

$$\begin{aligned} RHS^* &= ln(\frac{1}{1+\beta a_1}k) + \beta[a_0 + a_1 ln(\frac{\beta a_1 k}{1+\beta a_1})] \\ &= ln(k) - ln(1+\beta a_1) + \beta[a_0 + a_1\{ln(k) + ln(\beta a_1) - ln(1+\beta a_1)\}] \\ &= -(1+\beta a_1)ln(1+\beta a_1) + \beta[a_0 + a_1 ln(\beta a_1)] + (1+\beta a_1)ln(k) \end{aligned}$$

We know that $a_1 = (1+\beta a_1)$ and solving this gives us a_1:

$$\begin{aligned} a_1 &= (1+\beta a_1) \\ (1-\beta)a_1 &= 1 \\ a_1 &= \frac{1}{1-\beta} \end{aligned}$$

Now plugging this in for a_0 yields

$$\begin{aligned} a_0 &= -(1+\beta a_1)ln(1+\beta a_1) + \beta[a_0 + a_1 ln(\beta a_1)] \\ (1-\beta)a_0 &= -(\frac{1}{1-\beta})ln(1+\frac{\beta}{1-\beta}) + \frac{\beta}{1-\beta}ln(\frac{\beta}{1-\beta}) \\ a_0 &= \frac{-ln(1+\frac{\beta}{1-\beta}) + \beta ln(\frac{\beta}{1-\beta})}{(1-\beta)^2} \end{aligned}$$

Because this can be separated into an expression that follows $V(k) = a_0 + a_1 ln(k)$, our guess is verified.

d [20] Now suppose that they learn that with γ probability, a volcano on the island will erupt and their remaining capital becomes $k_{BOOM} = \delta k_t > 0$ with $\delta \in (0,1)$. Describe how their consumption path changes and how this depends on the value of γ. Answer:

Now, there is a probability γ that they will lose a share of their cake. We can write out their resulting recursive problem in the following way:

$$V(k) = \max_{c,k'} ln(c) + \beta[\gamma V_R(\delta k') + (1-\gamma)V(k')] \quad (9)$$

$$\text{s.t. } c + k' = k \quad (10)$$

Then the Euler Equation includes a gamble over two possible consumption states:

$$\frac{1}{c} = \beta[\gamma\frac{1}{c'(\delta k')} + (1-\gamma)\frac{1}{c'(k')}] \quad (11)$$

You can easily appeal to Jensen's Inequality and note that more resources implies higher average c, which implies lower marginal utility, and hence less consumption today. The easier way to note that this is a closed economy with no positive investment, which means we can write the present value of lifetime resources as

$$\hat{k}_0 = (1-\gamma)k_0 + \gamma\delta k_0 < k_0 \quad (12)$$

Because they have fewer resources to consume, they will necessarily consume less.