

# Macro II

Professor Griffy

UAlbany

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# Announcements

- ▶ Today: Solving heterogeneous agent models.
- ▶ Idea:
  - ▶ Solving these models is non-trivial.
  - ▶ Must consider the state of every agent in economy.
- ▶ Homework: start from the code on the cluster.

# Heterogeneous Agent Production Economy

- ▶ In a production economy, the agent's problem is given by

$$V(k, \epsilon; \psi) = u(c) + \beta E[V(k' \epsilon'; \psi')] \quad (1)$$

$$\text{s.t. } c + k' \leq (1 + r(K, L) - \delta)k + w(K, L)\epsilon \quad (2)$$

$$k' \geq \underline{k} \quad (3)$$

$$\epsilon \sim \text{Markov } P(\epsilon' | \epsilon) \quad (4)$$

$$\psi' = \Psi(\psi) \quad (5)$$

$$c \geq 0, k \geq 0, k_0 \text{ given} \quad (6)$$

- ▶  $\epsilon$  is a markov process that yields hours worked.
- ▶  $\Psi$  is an unspecified evolution of the aggregate state  $(k, \epsilon)$ .
- ▶ Prices are determined from the firm's problem

## Prices - The Firm's Problem

- ▶ How we handle prices determines the difficulty of this problem.
- ▶ In this economy, a single firm produces using labor (hours) and capital.

$$\Pi = \max_{K,L} F(K, L) - wL - rK \quad (7)$$

- ▶ This yields standard competitive prices for the rental rates.

# Stationary Recursive Competitive Equilibrium

- A stationary RCE is given by pricing functions  $r, w$ , a worker value function  $V(k, \epsilon; \psi)$ , worker decision rules  $k', c$ , a type-distribution  $\psi(k, \epsilon)$ , and aggregates  $K$  and  $L$  that satisfy
1.  $k'$  and  $c$  are the optimal solutions to the worker's problem given prices.
  2. Prices are formed competitively from the firm's problem.
  3. Consistency between aggregate evolution and individual decision rules:  $\psi$  is the stationary distribution implied by worker decision rules.
  4. Aggregates are consistent with individual policy rules:  
$$K = \int k d\psi, L = \int \epsilon d\psi$$

# Calibration

- ▶ Functions:

- ▶ Utility:  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$

- ▶ Production:  $F(K, L) = K^\alpha L^{1-\alpha}$

- ▶ Borrowing constraint:  $\underline{k} = 0$

- ▶  $\alpha = 0.36$ .

## Solving the Model: Market Clearing

- ▶ In equilibrium

$$K = \sum_k \sum_{\epsilon} k_s(k, \epsilon) \psi(k, \epsilon) \quad (8)$$

- ▶ where  $k_s$  is the supply of savings.
- ▶ What must the equilibrium prices satisfy?

$$r = F_K(K_D, L) \quad (9)$$

$$K_D(r) = K_S(r) \quad (10)$$

- ▶ Fixing  $K_D$  or  $r$  yields the other variable.
- ▶ Thus, one approach is to “guess” the equilibrium and iterate until we guess correctly.

## A Solution Technique: The Shooting Algorithm

- ▶ Guess  $r$ . Yields  $K_D$  and  $w$  from  $r = F_K(K_D, L)$  and  $w = F_L$ .
- ▶ Now, given this price, calculate the *individual* savings rule.
- ▶ Simulate the economy far enough into future to reach a steady-state distribution of capital.
- ▶ Check and see if  $K_D = K_S$ .
- ▶ If not, adjust guess of interest rate according to following:

$$r' = r + \lambda(K_D - K_S) \quad (11)$$

- ▶ where  $\lambda < 1$

## A Solution Technique: The Shooting Algorithm

- ▶ Adjusting interest rates:

$$r' = r + \lambda(K_D - K_S) \quad (12)$$

- ▶ If  $K_S > K_D$ : too much savings.
- ▶ Interest rate must fall to be in equilibrium.

## First iteration

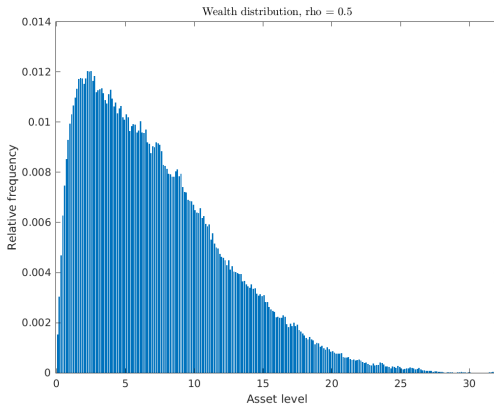
- ▶ Initial guess:
  - ▶  $r_0 = 0.03093$
- ▶ Three aggregates:
  1.  $K = 8.8342$
  2.  $L = 0.8582$
  3.  $\rightarrow r = F_K = 0.0204$
- ▶  $r - r_0 < \text{errtol}$ ?  $0.0309 - 0.0204$  too large.
- ▶ Algorithm: `fzero`  $\rightarrow$  pick local  $r_1$  and try again.

## Second iteration

- ▶ Initial guess:
  - ▶  $r_0 = 0.0308$
- ▶ Three aggregates:
  1.  $K = 1.4531$
  2.  $L = 0.9351$
  3.  $\rightarrow r = F_K = 0.1985$
- ▶  $r - r_0 < \text{errtol}$ ?  $0.0309 - 0.1935$  too large.
- ▶ Very sensitive to  $r_0$ !

# Converged Wealth Dist.

- ▶ Final wealth distribution after convergence:



## Another Solution technique: Root-Finding and Excess Demand

- ▶ Functionally, this is the same as what we just did.
- ▶ Suppose we solve household decision rules  $k$ , and  $r$ .
- ▶ Then, the excess demand function is

$$\Delta(r) = K_D(r) - K_S(r) \quad (13)$$

- ▶ Where we have solved  $K_D$  for many values of  $r$  and have an expression for  $K_S(r)$  (static firm optimization).
- ▶ Do one-dimensional root finding, i.e., find  $r^*$  such that

$$0 = \Delta(r^*) = K_D(r^*) - K_S(r^*) \quad (14)$$

# Aggregate Uncertainty



- ▶ What about aggregate uncertainty? The distribution is no longer stationary, hence no longer the equilibrium.

# Aggregate Uncertainty

- ▶ In a production economy, the agent's problem is given by

$$V(k, \epsilon; z, \psi) = u(c) + \beta E[V(k', \epsilon'; z', \psi')] \quad (15)$$

$$\text{s.t. } c + k' \leq (1 + r(z, K, L) - \delta)k + w(z, K, L)\epsilon \quad (16)$$

$$k' \geq \underline{k} \quad (17)$$

$$z' = \text{Markov}P(z'|z) \quad (18)$$

$$\epsilon \sim \text{Markov}P(\epsilon'|\epsilon, z') \quad (19)$$

$$\psi' = \Psi(\psi, z, z') \quad (20)$$

$$c \geq 0, k \geq 0, k_0 \text{ given}, z_0 \text{ given} \quad (21)$$

- ▶  $\epsilon$  is a markov process for employment  $\epsilon \in \{0, 1\}$
- ▶  $\Psi$  is an unspecified evolution of the aggregate state.
- ▶  $z$  *also* evolves as a markov process.
- ▶ Prices are determined from the firm's problem.

## Prices - The Firm's Problem

- ▶ How we handle prices determines the difficulty of this problem.
- ▶ In this economy, a single firm produces using labor (hours) and capital.

$$\Pi = \max_{K,L} zF(K, L) - wL - rK \quad (22)$$

- ▶ This yields standard competitive prices for the rental rates.

# Laws of Motion

- ▶ The future aggregate state enters the probability of employment.
- ▶ This means that it impacts **all** of the laws of motion:

$$z' = \text{Markov}P(z'|z) \quad (23)$$

$$\epsilon \sim \text{Markov}P(\epsilon'|\epsilon, z') \quad (24)$$

$$k' \leq (1 + r(z, K, L) - \delta)k + w(z, K, L)\epsilon - c \quad (25)$$

$$\psi' = \Psi(\psi, z, z') \quad (26)$$

- ▶ Because shocks to  $z$  change employment status and prices.

# Recursive Competitive Equilibrium

- ▶ An RCE is given by pricing functions  $r, w$ , a worker value function  $V(k, \epsilon, z; \psi)$ , worker decision rules  $k', c$ , a type-distribution  $\psi(k, \epsilon)$ , and aggregates  $K$  and  $L$  that satisfy
  1.  $k'$  and  $c$  are the optimal solutions to the worker's problem given prices.
  2. Prices are formed competitively from the firm's problem.
  3. Consistency between aggregate evolution and individual decision rules:  $\psi$  is the distribution implied by worker decision rules given the aggregate state.
  4. Aggregates are consistent with individual policy rules:  
$$K = \int k d\psi, L = \int \epsilon d\psi$$

# Type Distribution

- ▶ The type distribution is a problem.
- ▶ Each policy function and transition depends on the type distribution.
- ▶ But the type distribution is time-varying in response to aggregate shocks.
- ▶ Alternative: use a smaller number of moments that can be calculated quickly to characterize the type distribution.
- ▶ Like a “sufficient statistic” for the type distribution.

## Krusell and Smith (1998)

- ▶ Specify moments from the type distribution  $\gamma$  that approximate the type distribution.
- ▶ Then:  $\gamma' = \Gamma(\gamma, z, z')$ .
- ▶ Household predicts prices using  $\Gamma$  instead of  $\Psi$
- ▶ As long as this law of motion is reasonably accurate, this approximation will work.
- ▶ Krusell and Smith:
  - ▶ Pick first  $j$  moments of distribution over  $k, \epsilon$
  - ▶ i.e., mean, standard deviation,...
  - ▶ Use this as the law of motion.
- ▶ Use means:  $\ln(K') = \phi_0^z + \phi_1^z \ln(K)$

## Approximate problem

- ▶ In a production economy, the agent's problem is given by

$$V(k, \epsilon; z, K) = u(c) + \beta E[V(k', \epsilon'; z', K')] \quad (27)$$

$$\text{s.t. } c + k' \leq (1 + r(z, K, L) - \delta)k + w(z, K, L)\epsilon \quad (28)$$

$$k' \geq \underline{k} \quad (29)$$

$$z' = \text{Markov}P(z'|z) \quad (30)$$

$$\epsilon \sim \text{Markov}P(\epsilon'|\epsilon, z') \quad (31)$$

$$\ln(K') = \phi_0^z + \phi_1^z \ln(K) \quad (32)$$

$$c \geq 0, k \geq 0, k_0 \text{ given}, z_0 \text{ given} \quad (33)$$

- ▶ LLN  $\rightarrow$   $N$  known given  $z$ .
- ▶ Now: need aggregate capital and  $\phi_0^z, \phi_1^z$ .
- ▶ Note:  $\phi_0^z, \phi_1^z$  for each  $z$

# KS Solution Technique

► Algorithm:

1. Specify an initial forecasting function for  $K$ :  
 $\ln(K') = \phi_0^z + \phi_1^z \ln(K)$ . Pick initial values for  $\phi_0^z, \phi_1^z$
2. Tell household that the evolution of the aggregate state is given by  $\ln(K') = \phi_0^z + \phi_1^z \ln(K)$ . i.e., replace the previous constraint.
3. Use value function iteration on this problem to solve for optimal policy rules.
4. Simulate model forward to obtain  $K, z$  series. Drop first  $X$  number of observations.
5. Use OLS on  $K, z$  series to see if forecasting was correct  
 $||[\phi_0^z, \phi_1^z]' - \phi_0^z, \phi_1^z|| < \text{errtol}$
6. If not, update  $\phi_0^z, \phi_1^z$  between initial and estimates.

- Another way to think about this: You estimated the slope and intercept of  $K'$  on some series  $\{K_j, z_j\}_{j=1}^{j=t}$  and you are assessing its out of sample fit on  $\{K_j, z_j\}_{j=t+1}^T$

# KS Solution Technique

- ▶ Why does mean work?
- ▶ Linearity:

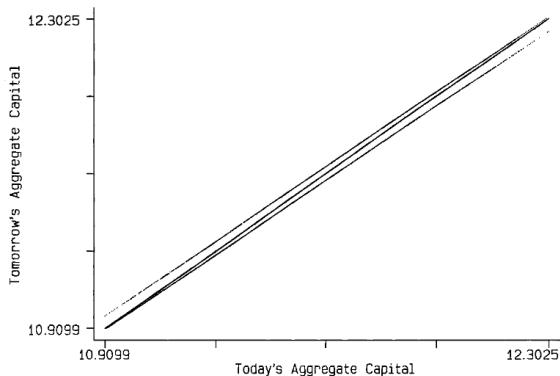


FIG. 1.—Tomorrow's vs. today's aggregate capital (benchmark model)

# What do they find?

- ▶ With  $\beta$  heterogeneity, can hit wealth dist.

TABLE 1  
DISTRIBUTION OF WEALTH: MODELS AND DATA

MODEL	PERCENTAGE OF WEALTH HELD BY TOP					FRACTION WITH WEALTH < 0	GINI COEFFICIENT
	1%	5%	10%	20%	30%		
Benchmark model	3	11	19	35	46	0	.25
Stochastic- $\beta$ model	24	55	73	88	92	11	.82
Data	30	51	64	79	88	11	.79

- ▶ What is heterogeneity in  $\beta$  a reduced-form for?

# Business Cycle Effects

- ▶ This model is built to handle stochastic shocks.
- ▶ How do heterogeneous agents respond over a business cycle?

TABLE 2  
AGGREGATE TIME SERIES

Model	Mean ( $k_t$ )	Corr( $c_t, y_t$ )	Standard Deviation ( $i_t$ )	Corr( $y_t, y_{t-4}$ )
Benchmark:				
Complete markets	11.54	.691	.031	.486
Incomplete markets	11.61	.701	.030	.481
$\sigma = 5$ :				
Complete markets	11.55	.725	.034	.551
Incomplete markets	12.32	.741	.033	.524
Real business cycle:				
Complete markets	11.56	.639	.027	.342
Incomplete markets	11.58	.669	.027	.339
Stochastic- $\beta$ :				
Incomplete markets	11.78	.825	.027	.459

That was easy



# Conclusion

- ▶ Today: solving heterogeneous agent models.
- ▶ Code to do this on the cluster.
- ▶ Start labor market frictions.