
AECO 701 Midterm Answers

A

Name:

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Cake Eating Problem. Tom Hanks is shooting a film where he is stranded on a small island. He and the film crew have two ($T = 2$) periods to complete the film. That is, they face the following problem:

$$\sum_{t=0}^1 \beta^t \ln(c_t) \quad (1)$$

Where $\beta \in (0, 1)$ and we have imposed log-utility ($u(c) = \ln(c)$). They do not value any island consumption after departing, so their continuation value is 0 ($V_T = 0$). The resource constraint is as follows:

$$c_t + k_{t+1} \leq k_t \quad (2)$$

where $c_t, k_{t+1} \geq 0$, and $k_0 > 0$ is given. Here, k_t is the filming capital remaining, which must be consumed by T . Their filming capital does not depreciate. Please keep in mind that this is a two-period model and include any terminal conditions.

a [20] Write the problem recursively and solve for the Euler Equation. [Answer:](#)

We can write the problem recursively as

$$V_t(k_t) = \max_{c_t, k_{t+1}} \ln(c_t) + \beta V_{t+1}(k_{t+1}) \quad (3)$$

$$\text{s.t. } c_t + k_{t+1} = k_t \quad (4)$$

$$V_2 = 0 \quad (5)$$

We find the Euler Equation by taking FOCs in consumption (c_t) and capital savings (k_{t+1}). Let λ_t be the multiplier on the corresponding optimization problem. Then

$$FOC[c_t] = \frac{1}{c_t} - \lambda_t = 0 \rightarrow \frac{1}{c_t} = \lambda_t \quad (6)$$

$$FOC[k_{t+1}] = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} - \lambda_t = 0 \quad (7)$$

$$\rightarrow \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} = \lambda_t \quad (8)$$

Our combined FOCs then yield

$$\frac{1}{c_t} = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \quad (9)$$

$$(10)$$

Now we (normally) need to use the envelope condition. Because this is a finite horizon, we don't actually need to for our application, but just for completeness:

$$\frac{\partial V_t}{\partial k_t} = \lambda_t \quad (11)$$

$$\rightarrow \frac{\partial V_{t+1}}{\partial k_{t+1}} = \lambda_{t+1} \quad \text{Envelope Push} \quad (12)$$

Then combining our FOCs with the Envelope Condition:

$$\frac{1}{c_t} = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \quad (13)$$

$$\frac{1}{c_t} = \beta \lambda_{t+1} \quad (14)$$

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} \quad (15)$$

b [20] Use the Euler Equation and the budget constraint to solve for explicit solutions for consumption and savings in each period (i.e., c_t, k_{t+1} in terms of parameters and initial conditions).

1. From the Euler Equation: $c_1 = \beta c_0$.
2. Since $c_1 = k_1$, we have $k_1 = \beta c_0$.
3. Substitute into the period 0 budget constraint: $c_0 + \beta c_0 = k_0 \implies c_0(1 + \beta) = k_0$.

Optimal Policy Functions:

- **Consumption:** $c_0^* = \frac{1}{1+\beta} k_0$
- **Savings:** $k_1^* = \frac{\beta}{1+\beta} k_0$

c [20] Does this problem satisfy the monotonicity requirement of Blackwell's Sufficient Conditions (below)? Show why or why not.

The monotonicity requirement of Blackwell's Sufficient Conditions for a contraction are given by the following: T is monotone if for $f(x) \leq g(x) \forall x \in X$, then

$$Tf(x) \leq Tg(x) \quad \forall x \in X \quad (16)$$

Answer:

This is just a straightforward application of Blackwell's Sufficient Conditions. A general proof is given below

Monotonicity:

Let $f(x) \leq g(x)$. Then

$$Tf(x) = h(x, y) + \beta f(x)$$

$$Tg(x) = h(x, y) + \beta g(x)$$

Taking the difference of these two yields:

$$\begin{aligned} Tf(x) - Tg(x) &= h(x, y) + \beta f(x) - [h(x, y) + \beta g(x)] \\ &= \beta(f(x) - g(x)) \end{aligned}$$

Since $f(x) \leq g(x)$, we know that $Tf(x) \leq Tg(x)$. Thus, the Bellman Operator is monotonic.

d [20] Use your value function from part (a) and policy functions from part (b) to show that the value function takes the following form: $V(k) = a_0 + a_1 \ln(k)$. Solve for a_0 and a_1 . **Answer:**

Using the Bellman Equation from (a) and the optimal policies from (b), we substitute c_0^* and k_1^* into the objective function to find the maximized value:

$$V_0(k_0) = \ln(c_0^*) + \beta \ln(k_1^*) \quad (17)$$

$$V_0(k_0) = \ln\left(\frac{1}{1+\beta}k_0\right) + \beta \ln\left(\frac{\beta}{1+\beta}k_0\right) \quad (18)$$

Using log properties:

$$V_0(k_0) = \underbrace{\left[\ln\left(\frac{1}{1+\beta}\right) + \beta \ln\left(\frac{\beta}{1+\beta}\right)\right]}_{a_0} + \underbrace{(1+\beta)}_{a_1} \ln(k_0) \quad (19)$$

Matching the form $V(k) = a_0 + a_1 \ln(k)$, we find:

- $a_1 = 1 + \beta$
- $a_0 = \ln\left(\frac{1}{1+\beta}\right) + \beta \ln\left(\frac{\beta}{1+\beta}\right)$

e [20] Write out the pseudo-code for solving this problem using “grid-search” (i.e., where the value, state, and choice functions are approximated as grids) on a computer, similar to HW1.

Step 1: Define the Grid Create a grid for the state variable $k \in [k_{min}, k_{max}]$ with N points. Let $K = \{k^{(1)}, k^{(2)}, \dots, k^{(N)}\}$.

Step 2: Solve the Terminal Period ($t = 1$) Since all capital is consumed in the final period ($c_1 = k_1$):

- For each $k \in K$:
- $V_1(k) = \ln(k)$

Step 3: Solve the Initial Period ($t = 0$) via Grid Search Initialize an empty vector V_0 of size N and a policy vector k'_1 of size N .

1. **Loop** over each possible initial state $k_i \in K$:
 - (a) **Loop** over each possible choice of next-period capital $k'_j \in K$:
 - i. **Check feasibility:** If $k'_j < k_i$:
 - ii. Calculate consumption: $c = k_i - k'_j$
 - iii. Calculate utility for this choice: $U(k_i, k'_j) = \ln(c) + \beta V_1(k'_j)$
 - iv. **Else:** Set $U = -\infty$ (infeasible)
 - (b) **Find Max:** $V_0(k_i) = \max_{k'_j} \{U(k_i, k'_j)\}$
 - (c) **Store Policy:** $k'_1(k_i) = \arg \max_{k'_j} \{U(k_i, k'_j)\}$

Step 4: Comparison The resulting vector V_0 is the numerical approximation of the $V_0(k_0) = a_0 + a_1 \ln(k_0)$ solved in part (d).— **Pseudo-code Summary:**

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Period 1 V1 = log(K)
Period 0 for i from 1 to N: for j from 1 to N: if K[j] < K[i]: c = K[i] - K[j] ValCandidate[j] =
log(c)+beta*V1[j]else : ValCandidate[j] = -infV0[i] = max(ValCandidate)Policyk1[i] = K[argmax(ValCandidate)]

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